Text-Book

ON

Integral Calculus

AND

Elementary Differential Equations

By

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PREFACE TO THE FIRST EDITION

This book is intended to serve as a companion volume to my Text-book on Differential Calculus and has been written to meet the requirements of the B.A. and B.Sc. students of Indian Universities. The treatment of the subject is in keeping with the modern theory of functions, but is at the same time simple. The integration of the ordinary functions has been systematically and fully dealt with. The book has been divided into sections according to the function to be integrated and not according to the methods to be employed, as the beginner is not able to say which method will be applicable in a given case.

All the elementary methods of integration have been dealt with in the first chapter, so that after studying it the student may have no difficulty in following discussions in physics or applied mathematics where integration is involved. The chapter on applications deals with the problems of finding centres of gravity, centres of pressure and moments of inertia, and will be found fairly complete. In differential equations only equations of the first order and linear equations with constant coefficients have been fully dealt with. A few other types have been briefly discussed, but equations with non-constant coefficients of an order higher than the first and partial equations have been entirely omitted. The historical and biographical notes in the book will, it is hoped, prove interesting. The information they supply should be supplemented by the historical sketch given in the Text-Book on Differential Calculus.

As in the case of the companion volume, the present work also contains just a little more than is necessary for the usual course. No hesitation need be felt, therefore, in omitting some of the articles. The number of exercises also will be found to be ample. Of these some are original, many have been taken from the examination papers of the various universities, and others are such as are common to practically all text-books on the subject. Often, where a particular university is mentioned, the object is to indicate what types of questions are generally set in the examinations, rather than to indicate the original authorship of the problem.

I am greatly indebted to Mr. Harish Chandra Gupta M.Sc., a former pupil of mine, now lecturer in Mathematics, Christ Church College, Cawnpore, who has read with great care the proofs of the whole of the book, verified most of the examples, and made many valuable suggestions. My thanks are also due to my colleagues, Dr. B. N. Prasad, Ph.D.,D.Sc., and Mr. R. N. Choudhuri, B.A. (Cantab.), to my friends Dr. R. S. Varma, D.Sc., and Mr. J. L. Sharma, M.A., and to several students of my B.Sc., and post-graduate classes, who have all very generously helped me in reading the proofs or verifying the examples or have offered valuable criticism and advice.

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PREFACE TO THE EIGHTH EDITION

This edition differs the earlier ones only in minor points.

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1·11. Definitions. *If* $\frac{d}{dx}F(x)=f(x)$, *we say that* $F(x)$ *is an integral of* $f(x)$. We write

$$\int f(x)\,dx=F(x).$$

Thus $$\int \cos x\,dx=\sin x,$$

because $$d\sin x/dx=\cos x.$$

Similarly $$\int x^2\,dx=\tfrac{1}{3}x^3,$$

$$\int e^x\,dx=e^x;\ \text{etc.}$$

Now, if $$dF(x)/dx=f(x),$$

and C is an arbitrary constant, we also have

$$d\{F(x)+C\}/dx=f(x).$$

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Now, if $$dF(x)/dx=f(x),$$

and C is an arbitrary constant, we also have

$$d\{F(x)+C\}/dx=f(x).$$

It follows that, *if* $\mathbf{dF(x)/dx=f(x)}$,

then $$\int \mathbf{f(x)\,dx=F(x)+C.}$$

Just as there are two square-roots, $+\sqrt{x}$ and $-\sqrt{x}$, of a given number x, whereas there is only one square of x, and again there are an infinite number of values of $\sin^{-1}x$ although $\sin x$ has only one value, similarly the integral of $f(x)$ has an infinite number of values, obtained by giving to C in $F(x)+C$ different values. We cannot find a more general value for the integral than $F(x)+C$. Hence we call it *the* integral of $f(x)$.

The process of finding the integral is called *integration.* We are said to integrate $f(x)$ when we find the integral of $f(x)$. If there is any likelihood of doubt as to which symbol is the variable, we make it clear by saying some such thing as "integrate $f(x)$ *with regard to* (or *with respect to*) x".

The function which is integrated is called the *integrand.*

We notice that

$$\frac{d}{dx}\left\{\int f(x)\,dx\right\}=\frac{d}{dx}\left\{F(x)+C\right\}=f(x),$$

i.e., $$\frac{d}{dx}\int f(x)\,dx=f(x),$$

showing that differentiation and integration are inverse processes.

$\int f(x)\,dx$ is read as "integral of $f(x)\,dx$." The symbols $\int$ and dx may be regarded as something like a pair of brackets between which the function to be integrated is inserted. $\int$ and dx, taken separately, must be regarded as having no meaning. When the variable is something other than x we write that variable in place of x in the dx also. Thus

$$\int \lambda\,d\lambda=\tfrac{1}{2}\lambda^2+C,$$

and $$\int \lambda\cos x\,d\lambda=\tfrac{1}{2}\lambda^2\cancel{\cos} x+C,$$

whereas $$\int \lambda\cos x\,dx=\lambda\sin x+C.$$

In the second example the variable is λ, and x is regarded as a constant; in the third x is the variable and λ is regarded as a constant. In the second example we are able to assert that

$$\int \lambda \cos x \, d\lambda = \tfrac{1}{2}\lambda^2 \cos x + C,$$

because we know that $d(\frac{1}{2}\lambda^2 \cos x)/d\lambda = \lambda \cos x$.

We have said above that $\int$ and dx may be regarded as something like a pair of brackets between which the function to be integrated is inserted. But very often, when the function to be integrated is a fraction, the dx is written in the numerator, and the factor 1, if it occurs, is dropped. Thus we often write

$$\int \frac{\sin x \, dx}{\log x} \text{ instead of } \int \frac{\sin x}{\log x} dx$$

and

$$\int \frac{dx}{x} \text{ instead of } \int \frac{1}{x} dx.$$

1·12. The constant of integration. The general practice is to omit the constant of integration C.

Thus, almost invariably, we write

$$\int \cos x \, dx = \sin x,$$

instead of

$$\int \cos x \, dx = \sin x + C.$$

When this practice is adopted, it must be borne in mind that two integrals of a function may differ by a constant. Thus x^2+1 and x^2+2 are both integrals of $2x$, because

$$d(x^2+1)/dx = 2x,$$

and also

$$d(x^2+2)/dx = 2x.$$

But x^2+1 and x^2+2 are not equal. They differ by a constant.

Sometimes this fact is not quite apparent. Thus, because

$$d(\sin^{-1} x)/dx = 1/\sqrt{(1-x^2)},$$

and also $$d(-\cos^{-1} x)/dx = 1/\sqrt{(1-x^2)},$$

it follows, when we omit the constant of integration, that

$$\int \frac{1}{\sqrt{(1-x^2)}}\, dx$$

is equal to $\sin^{-1} x$ and also to $-\cos^{-1} x$. We must not infer from this that $\sin^{-1} x$ and $-\cos^{-1} x$ are equal. The correct inference is that, when the arbitrary constants C_1 and C_2 are properly adjusted,

$$\sin^{-1} x + C_1 = -\cos^{-1} x + C_2.$$

It is easy to see that this, in fact, is the case. If we take $C_2 - C_1$ to be equal to $\frac{1}{2}\pi$, we see at once that the above is true; for, by trigonometry,

$$\sin^{-1} x = \tfrac{1}{2}\pi - \cos^{-1} x.$$

The arbitrary constant of integration may be imaginary, i.e., may involve $\sqrt{(-1)}$. Very often such a constant is added to make the result real.

Thus $\displaystyle\int \frac{1}{x^2-a^2}\, dx$ is taken to be

$$\frac{1}{2a} \log \frac{x-a}{x+a} \quad \text{or} \quad \frac{1}{2a} \log \frac{a-x}{a+x},$$

whichever is real. The student will easily see that these two values of the integral differ by $i\pi/2a$.

From now onwards we also shall omit the constant of integration, unless there are special reasons for not doing so.

1·13. Standard forms. The integrals of several simple functions follow at once from the standard results in differential calculus. Thus we have

$\int x^n\,dx = x^{n+1}/(n+1)$

$\int (1/x)\,dx = \log x$

$\int e^x\,dx = e^x$

$\int a^x\,dx = a^x/\log a$

$\int \sin x\,dx = -\cos x$

$\int \cos x\,dx = \sin x$

$\int \sec^2 x\,dx = \tan x$

$\int \operatorname{cosec}^2 x\,dx = -\cot x$

$\int \sec x \tan x\,dx = \sec x$

$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x$

$\int \{1/\sqrt{(1-x^2)}\}\,dx = \sin^{-1} x$

$\int \{1/(1+x^2)\}\,dx = \tan^{-1} x$

$\int \{1/x\sqrt{(x^2-1)}\}\,dx = \sec^{-1} x$

$\int \{1/\sqrt{(2x-x^2)}\}\,dx = \operatorname{vers}^{-1} x$

$\int \cosh x\,dx = \sinh x$

$\int \sinh x\,dx = \cosh x$

etc.

A more complete list will be given later. It should be noticed that in the first formula n should not be equal to -1. This formula can be expressed more conveniently in words as follows :

To find the integral of x^n, increase the index of x by unity and divide by the new index.

1·21. Integral of the product of a constant and a function.

Let $$\int f(x)\,dx = F(x).$$

Then, by definition, $dF(x)/dx = f(x)$.

Now, by diff. calculus, $d\{aF(x)\}/dx = af(x)$.

Hence by the definition of the integral,

$$\int \{af(x)\}\,dx = aF(x) = a\int f(x)\,dx,$$

i.e., *the integral of the product of a constant and a function is equal to the product of the constant and the integral of the function.*

Ex. 1. $\int 3\cos x\,dx = 3\int \cos x\,dx = 3\sin x.$

Ex. 2. $\int ae^x\,dx = a\int e^x\,dx = ae^x.$

1·22. Integral of a sum.

Let $\int f_1(x)\,dx = F_1(x)$ and $\int f_2(x)\,dx = F_2(x)$.

Then $d\{F_1(x)+F_2(x)\}/dx = dF_1(x)/dx + dF_2(x)/dx$
$= f_1(x) + f_2(x).$

It follows, from the definition of the integral, that

$$\int \{f_1(x)+f_2(x)\}\,dx = F_1(x)+F_2(x)$$
$$= \int f_1(x)\,dx + \int f_2(x)\,dx.$$

It is evident that this method is applicable also to any expression consisting of a finite number of functions connected by the signs $+$ or $-$. Thus

$$\int \{\mathbf{f}_1(\mathbf{x}) \pm \mathbf{f}_2(\mathbf{x}) \pm \dots \pm \mathbf{f}_n(\mathbf{x})\}\,\mathbf{dx}$$
$$= \int \mathbf{f}_1(\mathbf{x})\,\mathbf{dx} \pm \int \mathbf{f}_2(\mathbf{x})\,\mathbf{dx} \pm \dots \pm \int \mathbf{f}_n(\mathbf{x})\,\mathbf{dx}.$$

Ex. 1. $\int (\cos x + x^n)\,dx = \int \cos x\,dx + \int x^n\,dx$
$= \sin x + x^{n+1}/(n+1).$

Ex. 2. $\int (2\sin x + 1/x)\,dx = -2\cos x + \log x.$

EXAMPLES

Write down the integrals of the following :

1. x^7, $3x^5$, $2x^{-3}$.
2. $3x+4x^2$, $(5x+7)/x$.
3. $e^x+2\sin x-3\cos x$.
4. $5\cos x+2\sec^2 x-10$.
5. $10^x+3e^x+x^3$.
6. $2/(1+x^2)+3a^x$.
7. $6/\sqrt{(1-x^2)}+3\sec^2 x$.
8. $\sec x\tan x-5\operatorname{cosec}^2 x$.
9. $2\cos x/(3\sin^2 x)+1$.
10. $1+x+x^2/2!+x^3/3!+\ldots$.
11. ax^3+bx^2+cx+d.
12. $a/x^2+b/x+c$.
13. $(4x^2+3x+2)/x^3$.
14. $(2x^3+3x-7)x^{-2/3}$.
15. $(x^2+8)^2/x^4$.
16. $(x+a)^3/\sqrt{x}$.

1·3. Methods of integration. Corresponding to the various rules in the differential calculus for differentiating sums, products and functions of functions, we have more or less similar rules in the integral calculus. These give rise to the following methods of integration:

(i) *Substitution*, corresponding to the rule for differentiating a function of a function.

(ii) *Integration by parts*, corresponding to the rule for differentiating a product of two functions.

(iii) *Decomposition into a sum.*

(iv) *Successive reduction.*

The various rules in the differential calculus enable us to differentiate almost any combination of the various ordinary functions. But it is not so with integration. In fact the integrals of some even fairly simple functions cannot be found, i.e., cannot be expressed in terms of the functions which are known to the student at this stage. For example, $\int\sqrt{(ax^3+bx^2+cx+k)}\,dx$ is not expressible in terms of the algebraic, exponential, logarithmic, or circular functions, except when the coefficients a, b, c, k have very special values.

1·31. Substitution.

Since $\dfrac{d}{dx}F\{\phi(x)\}=f\{\phi(x)\}\,.\,\phi'(x)$,

where the functions F and f are related by the equation

$$dF(t)/dt-f(t),$$

it follows that

$$\left.\begin{aligned}\int f\{\phi(x)\}\phi'(x)\,dx&=F\{\phi(x)\},\\ \textit{where}\qquad \int f(t)\,dt&=F(t).\end{aligned}\right\}\quad . \quad . \quad (1)$$

In applying this formula it is more convenient to adopt the following rule :

To evaluate $\displaystyle\int f\{\phi(x)\}\,\phi'(x)\,dx,$

$$\left.\begin{aligned}\textit{put}\qquad \boldsymbol{\phi(x)}&=\mathbf{t},\\ \textit{and}\qquad \boldsymbol{\phi'(x)\,dx}&=\mathbf{dt},\end{aligned}\right\}\quad . \quad . \quad (2)$$

where $\phi'(x)$ is the differential coefficient of $\phi(x)$.

These substitutions will evidently give us

$$\int f(t)\,dt,$$

in place of $\int f\{\phi(x)\}\,\phi'(x)\,dx$ and this when evaluated will give us $F(t)$, i.e , $F\{\phi(x)\}$, showing that the substitution (2) is equivalent to the formula (1).

In the examples which the student gets for integration, the functions $f\{\phi(x)\}$ and $\phi'(x)$ are generally so mixed up that the substitution $\phi(x)=t$ has often to be obtained by guess, rather than in accordance with some plan.

Ex. Evaluate $\int x \cos x^2\, dx.$

Putting $x^2 = t$ and consequently $2x\,dx = dt$,

i.e.,
$$x\,dx = \tfrac{1}{2}dt,$$

we get
$$\int x \cos x^2\, dx = \tfrac{1}{2}\int \cos t\, dt = \tfrac{1}{2}\sin t = \tfrac{1}{2}\sin x^2.$$

NOTE. 1. It is not necessary to attach any meaning to the dx or the dt in the equation which occurs in the working rule, viz., in the equation $\phi'(x)\,dx = dt$. The working rule may be regarded as a convenient method of passing from $\int f\{\phi(x)\}\,\phi'(x)\,dx$ to $\int f(t)\,dt$.

2. Another form of the theorem is the following :

$$\int \mathbf{f(x)\,dx} = \int \mathbf{f(x)\,\frac{dx}{dt}\,dt}. \qquad (3)$$

To obtain this, write t for $\phi(x)$ and dt/dx for $\phi'(x)$ in the formula (1). We get

$$\int f(t)\,\frac{dt}{dx}\,dx = F(t) = \int f(t)\,dt.$$

Interchanging x and t, we obtain the required result at once.

With the help of this form of the theorem we can avoid using the symbols dx and dt to which no meaning has been assigned. Thus if we want to evaluate

$$\int x \cos x^2\, dx,$$

we put $x^2 = t$, so that $2x\,dx/dt = 1$. Hence

$$\int x \cos x^2\, dx = \int x \cos x^2\,\frac{dx}{dt}\,dt, \text{ by formula (3)},$$

$$= \tfrac{1}{2}\int \cos t\,dt, \text{ on writing } x^2 = t,$$

$$= \tfrac{1}{2}\sin t = \tfrac{1}{2}\sin x^2.$$

1·32. Typical examples of the method of substitution.

(i) *Functions of a linear function of x.* To integrate a function of $ax+b$, put $ax+b=t$ and therefore

$$dx=(1/a)\,dt.$$

Then
$$\int \sin(ax+b)\,dx=\int(\sin t)(1/a)\,dt$$
$$=(1/a)\int \sin t\,dt=-(1/a)\cos t.$$

i.e.,
$$\int \mathbf{sin\,(ax+b)\,dx}=-\frac{\mathbf{1}}{\mathbf{a}}\mathbf{cos\,(ax+b).}$$

Similarly
$$\int \mathbf{(ax+b)^n\,dx}=\frac{\mathbf{(ax+b)^{n+1}}}{\mathbf{(n+1)a}},$$
$$\int \frac{\mathbf{1}}{\mathbf{ax+b}}\,\mathbf{dx}=\frac{\mathbf{1}}{\mathbf{a}}\mathbf{log\,(ax+b);}$$
$$\int \mathbf{e}^{ax}\,\mathbf{dx}=\frac{\mathbf{1}}{\mathbf{a}}\mathbf{e}^{ax},$$
$$\int \mathbf{cos\,(ax+b)\,dx}=\frac{\mathbf{1}}{\mathbf{a}}\mathbf{sin\,(ax+b),}$$
$$\int \mathbf{sec^2\,(ax+b)\,dx}=\frac{\mathbf{1}}{\mathbf{a}}\mathbf{tan\,(ax+b);}\ \text{etc.}$$

In general, if $\int f(x)\,dx=F(x)$, then

$$\int f(ax+b)\,dx=\frac{1}{a}F(ax+b).$$

For, putting $ax+b=t$, and $a\,dx=dt$,

i.e., $dx=(1/a)\,dt$, we have

$$\int f(ax+b)\,dx=\int f(t)\,\frac{dt}{a}=\frac{1}{a}F(t)=\frac{1}{a}F(ax+b).$$

(ii) *Functions involving $a^2 \pm x^2$.* When the corresponding integral involving $1 \pm x^2$ is known, the proper substitution is to put $x = at$. Thus

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2+a^2t^2}\,a\,dt$$

$$= \frac{1}{a}\int \frac{1}{1+t^2}\,dt = \frac{1}{a}\tan^{-1} t,$$

i.e.,
$$\boldsymbol{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)}.$$

Similarly
$$\boldsymbol{\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)},$$

$$\boldsymbol{\int \frac{1}{x\sqrt{(x^2-a^2)}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)}$$

$$\boldsymbol{\int \frac{1}{\sqrt{(2ax-x^2)}}\,dx = \mathrm{vers}^{-1}\left(\frac{x}{a}\right)}.$$

The formulae in heavy type are standard results and can be freely used to write down the result of integration without making any substitution.

EXAMPLES

Integrate with respect to x :

1. x^3, $(x+2)^3$, $(2x+3)^3$, $(3x-2)^3$, $(2-3x)^3$, $(ax+b)^3$.
2. $\frac{1}{x^3}$, $\frac{1}{(x+3)^3}$, $\frac{1}{(3x+4)^3}$, $\frac{1}{(8-3x)^3}$, $\frac{1}{(a+bx)^3}$, $\frac{1}{(a-bx)^3}$.
3. $\sqrt{x}$, $\sqrt{(x+4)}$, $\sqrt{(2x+3)}$, $\sqrt{(3-2x)}$, $\sqrt{(a+bx)}$.
4. $\sqrt{x^3}$, $\sqrt{(x+5)^3}$, $\sqrt{(5-x)^3}$, $\sqrt{(2x-7)^3}$, $\sqrt{(ax-b)^3}$.
5. $1/x$, $1/(x+1)$, $1/(2x+1)$, $1/(1-x)$, $1/(ax+b)$, $1/(b-ax)$.
6. e^{2x}, e^{2x+3}, e^{3-2x}, a^{3x}, a^{4x-5}, 10^{6x}.
7. $\sin 2x$, $\sin \frac{1}{2}x$, $\sin mx$, $\sin (x/m)$, $\sin (2x+\frac{1}{4}\pi)$.

8. $\dfrac{1}{\sqrt{\{1-x^2\}}}$, $\dfrac{1}{\sqrt{\{1-4x^2\}}}$, $\dfrac{1}{\sqrt{\{1-(2x-1)^2\}}}$, $\dfrac{1}{\sqrt{\{1-(3x-2)^2\}}}$, $\dfrac{1}{\sqrt{\{1-(ax+b)^2\}}}$.

9. $\dfrac{1}{1+4x^2}$, $\dfrac{1}{1+\frac{1}{4}x^2}$, $\dfrac{1}{1+a^2x^2}$, $\dfrac{1}{1+(ax+b)^2}$, $\dfrac{1}{1+x^2/a^2}$, $\dfrac{1}{a^2+x^2}$, $\dfrac{1}{7+x^2}$, $\dfrac{1}{7+4x^2}$, $\dfrac{1}{a^2+b^2x^2}$.

10. $\cos 3x$, $\operatorname{cosec}^2 5x$, $\sec^2 2x$, $\cos(3x+4)$, $\sec^2(7x+2)$.

11. $1/(9x+1)$, $\cos \frac{1}{2}x$, $\sec^2 3x$, $\tan 3x \sec 3x$, $\sin 2x/\cos^2 2x$.

12. $\sinh 2x$, $\cosh(ax+b)$, $\operatorname{sech}^2(3x-7)$, $\operatorname{cosech}^2(a-bx)$.

13. $4\sin 2x+8\sin 3x$.

14. $5a^x+6a\cos(5x+2)$.

15. $2(x-\frac{1}{2})^{-3}-3(x-\frac{1}{2})^{-2}$.

16. $\cos\frac{1}{3}(a-x)+\sin\frac{1}{3}(a+x)$.

1·33. Typical examples of the method of substitution (*continued*).

(iii) *Functions of x^n.* Functions of x^n multiplied by x^{n-1} can be simplified for purposes of integration by putting $x^n=t$.

Thus $\int x^2\sin x^3\,dx=\frac{1}{3}\int \sin t\,dt$, putting $x^3=t$ and $3x^2\,dx=dt$, i.e., $x^2\,dx=\frac{1}{3}dt$,

$$=-\tfrac{1}{3}\cos t=-\tfrac{1}{3}\cos x^3.$$

$$\int\frac{2x^3}{4+x^8}\,dx=\frac{2}{4}\int\frac{dt}{2^2+t^2},\text{ putting } x^4=t,\text{ and } 4x^3\,dx=dt,\text{ i.e., } x^3\,dx=\tfrac{1}{4}dt,$$

$$=\tfrac{1}{2}\cdot\tfrac{1}{2}\tan^{-1}\tfrac{1}{2}t,\text{ by § 1·32 (ii)},$$

$$=\tfrac{1}{4}\tan^{-1}\tfrac{1}{2}x^4.$$

(iv) *Powers of functions.* Any power of a function multiplied by the differential coefficient of the

function can be immediately integrated by substituting t for the function.

Thus $\int \sin^2 x \cos x \, dx = \int t^2 \, dt$, by putting $\sin x = t$ and $\cos x \, dx = dt$,

$$= \tfrac{1}{3} t^3 = \tfrac{1}{3} \sin^3 x.$$

Examples

Integrate with respect to x:

1. xe^{x^2}, xa^{x^2}, $x \sin x^2$.
2. $x^2/(1+x^6)$, [*Del.*, '62] $x\sqrt{(1+x^2)}$, $x(2+x^2)^{3/2}$.
3. $x(a^2+x^2)^n$, $x/\sqrt{(a^2-x^4)}$, $x/(a^2+x^4)$.
4. $x^4(x^5-1)^{1/3}$, $x^2/\sqrt{(a^3+x^3)}$, $x^2/\sqrt{(a^3-x^3)}$.
5. $x^{n-1}/(a+bx^n)$, $x^{n-1}/(4+x^n)$, $x^{p-1}/\sqrt{(2-x^p)}$.
6. $\sin^2 x \cos x$, $(\log x)^2/x$, $\cos^2 x \sin x$.
7. $\tan^2 x \sec^2 x$, $\cos^2 x/\sin^4 x$, $(\sin^{-1} x)^2/\sqrt{(1-x^2)}$.
8. $(\sin x)^{-2} \cos x$, $\sec^2 x \sin x$, $\operatorname{cosec}^n x \cos x$.
9. $\tan^p x \sec^2 x$, $(\cot^{-1} x)/(1+x^2)$, $\sec^{-1} x/x\sqrt{(x^2-1)}$.
10. $(1+\log x)^3/x$, $e^x(a+be^x)^n$, $(1+\sin x)^2 \cos x$.
11. $(a+b \sin x)^p \cos x$, $(a-b \tan x)^q \sec^2 x$, $(a+b \sin^{-1} x)^m/\sqrt{(1-x^2)}$.
12. $\dfrac{4x^2}{\sqrt{(1-x^6)}}$, $\dfrac{4x^2}{1+x^6}$, $\dfrac{4x^2}{\sqrt{(3-x^6)}}$, $\dfrac{4x^2}{2+3x^6}$.
13. $\dfrac{1}{x\sqrt{(x^2-1)}}$, $\dfrac{1}{x\sqrt{(x^4-1)}}$, $\dfrac{1}{x\sqrt{(x^4-4)}}$.
14. $\dfrac{x}{\sqrt{(1-2x^4)}}$, $\dfrac{3x}{1+2x^4}$, $\dfrac{1}{7x\sqrt{(2x^4-1)}}$.
15. $\dfrac{(\sin^{-1} x)^3}{\sqrt{(1-x^2)}}$, $\dfrac{(\tan^{-1} x)^4}{1+x^2}$, $\dfrac{(\sec^{-1} x)^5}{x\sqrt{(x^2-1)}}$, $\dfrac{(\operatorname{vers}^{-1} x)^6}{\sqrt{(2x-x^2)}}$.

1·34. Typical examples of the method of substitution (*continued*).

(v) *Fraction in which the numerator is the differential coefficient of the denominator.* Putting the denominator equal to t, we see that

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{dt}{t} = \log t = \log f(x),$$

i.e., *the integral of a fraction in which the numerator is the differential coefficient of the denominator is equal to the logarithm of the denominator.*

Thus $$\int \frac{\sin x}{\cos x}\,dx = -\int -\frac{\sin x}{\cos x}\,dx = -\log \cos x,$$

i.e., $$\boldsymbol{\int \tan x\,dx = -\log \cos x} = \log \sec x.$$

Similarly $$\boldsymbol{\int \cot x\,dx = \log \sin x},$$

$$\int \frac{x}{x^2+1}\,dx = \tfrac{1}{2}\int \frac{2x}{x^2+1}\,dx = \tfrac{1}{2}\log(x^2+1)\,;\ \text{etc.}$$

(vi) *Other cases.* The method of substitution is the most powerful method of integration and a great variety of substitutions is used. Experience alone will enable the student to think out a suitable substitution, but some more standard substitutions will be dealt with in the succeeding chapters. Sometimes two or more substitutions in succession are used.

Ex. Integrate $x^2(\tan^{-1}x^3)/(1+x^6)$.

$$\int \frac{x^2\tan^{-1}x^3}{1+x^6}\,dx = \tfrac{1}{3}\int \frac{\tan^{-1}t}{1+t^2}\,dt, \text{ putting } x^3=t, \text{ and } 3x^2\,dx=dt,$$

$$= \tfrac{1}{3}\int u\,du, \text{ putting } \tan^{-1}t=u \text{ and } \{1/(1+t^2)\}\,dt=du,$$

$$= \tfrac{1}{3}\cdot\tfrac{1}{2}u^2 = \tfrac{1}{6}(\tan^{-1}t)^2 = \tfrac{1}{6}(\tan^{-1}x^3)^2.$$

EXAMPLES

Integrate with respect to x :

1. $\dfrac{3x^2}{x^3+1}$, $\dfrac{2x+3}{x^2+3x+2}$, $\dfrac{ax+b}{ax^2+2bx+c}$, $\dfrac{ax^{n-1}}{x^n+b}$.

2. $\dfrac{e^x}{e^x+1}$, $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ [*Alld.*, '59] $\dfrac{10x^9+10^x.\log_e 10}{10^x+x^{10}}$.

3. $\dfrac{\operatorname{cosec}^2 x}{1+\cot x}$, $\dfrac{1}{(1+x^2)\tan^{-1}x}$, $\dfrac{1}{\sqrt{(1-x^2)}\cos^{-1}x}$.

4. $\dfrac{1}{x\log x}$, $\dfrac{\sin x}{a+b\cos x}$, $\dfrac{\sin x\cos x}{a\cos^2 x+b\sin^2 x}$.

5. $\dfrac{\sin x}{1+\cos^2 x}$, $\dfrac{\sin x}{a^2+b^2\cos^2 x}$, $\dfrac{\sin(a+b\log x)}{x}$.

6. $\dfrac{\sin x}{\sqrt{(a^2-\cos^2 x)}}$, $\dfrac{1}{x\cos^2(1+\log x)}$, $\dfrac{1}{x(1+\log x)^m}$.

7. $\dfrac{e^{\arctan x}}{1+x^2}$ [*Cal.*, '39]; $\dfrac{\sin\sqrt{x}}{\sqrt{x}}$, $\dfrac{a^{\arccos x}}{\sqrt{(1-x^2)}}$, $\dfrac{\sin(\tan^{-1}x)}{1+x^2}$.

8. $x\cos^3 x^2\sin x^2$, $x^3\tan^4 x^4\sec^2 x^4$.

1·35. Integral of the product of two functions. Integration by parts. If $f(x)$ and $\phi(x)$ be two functions of x we know that

$$d\{f(x).\phi(x)\}/dx=f(x).\phi'(x)+f'(x).\phi(x).$$

Hence, by definition and § 1·22,

$$f(x).\phi(x)=\int f(x).\phi'(x)\,dx+\int f'(x).\phi(x)\,dx,$$

or $$\int f(x).\phi'(x)\,dx=f(x).\phi(x)-\int f'(x).\phi(x)\,dx.$$

To write the result in a more symmetrical form, replace $f(x)$ by $f_1(x)$ and write $f_2(x)$ for $\phi'(x)$. Then for $\phi(x)$ we shall have to write $\int f_2(x)\,dx$. The above equation then becomes

$$\int f_1(x)f_2(x)\,dx = f_1(x)\int f_2(x)\,dx - \int\{f_1'(x)\int f_2(x)\,dx\}\,dx,$$

i.e., *the integral of the product of two functions*

Imp

=first function × integral of second −integral of {diff. coeff. of first × integral of second}.

Integration with the help of this rule is called *integration by parts*. The success of the method depends upon choosing the first function in such a way that the second term on the right-hand side may be easy to evaluate. Often it makes much difference which function in the product is regarded as the first function. So care must be taken in choosing the first function.

It is important to note that :

(i) Unity may be taken in certain cases as one of the factors.

(ii) The formula of integration by parts can be applied more than once if necessary.

(iii) If the integral on the right hand side reverts to the original form, the value of the integral can be immediately inferred by transposing the former to the left-hand side.

Ex. 1. $\int x\cos x\,dx = x\,.\,\sin x - \int 1\,.\,\sin x\,dx$

$$= x\sin x + \cos x.$$

Ex. 2. $\int \log x\,dx = \int (\log x)\,.\,1\,dx$ [*Ujjain*, '62]

$$= (\log x)\,.\,x - \int \frac{1}{x}\,.\,x\,dx = x\log x - x = x\log(x/e).$$

Ex. 3. $\int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx$

$$= x^2\sin x - 2\{x(-\cos x) - \int 1\,.\,(-\cos x)\,dx\}$$

$$= x^2\sin x + 2x\cos x - 2\sin x.$$

Ex. 4. Integrate $e^x \sin x$.

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx$$
$$= -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx.$$

Transposing and dividing by 2,

$$\int e^x \sin x\, dx = \tfrac{1}{2} e^x (\sin x - \cos x).$$

EXAMPLES

Integrate :

1. $x \log x$ [*Madras*, '60]; $(\log x)/x^2$; $x^n \log x$. [*Osm.*, '60]
2. xe^x, xe^{ax}, $x \sinh x$.
3. $x \cos x$, $x \cos nx$, $x \operatorname{cosec}^2 ax$.
4. $\tan^{-1} x$ [*Gorakhpur*, '60]; $\cot^{-1} x$; $\sin^{-1} x$. [*Gauhati*, '49]
5. $x^2 \sin x$ [*Ujjain*, '62]; $x^2 \cos 2x$; $x^2 e^{mx}$. [*Baroda*, '59]
6. $(\log x)^2$ [*Andhra*, 1950]; $x^n (\log x)^2$; $x^3 e^{-x}$. [*Utkal*, '56]
7. $e^x \cos x$, $e^{2x} \sin x$ [*Andhra*, 1962] $e^{3x} \cos 2x$.

1·36. Breaking up the integrand into a sum. We have proved before (§ 1·22) that the integral of the sum of a number of functions is equal to the sum of the integrals of the functions. An important method of integration consists in breaking up a given function into the sum of a number of suitable functions and then applying this theorem. This method is particularly useful in the case of rational algebraic fractions and certain trigonometrical products. (See Chaps. II and IV.) Thus, we know that

$$\frac{1}{x^2 - a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right).$$

Hence $$\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a}\left\{\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right\}$$
$$= (1/2a)\{\log (x-a) - \log (x+a)\},$$

i.e., $$\int \frac{\mathbf{1}}{\mathbf{x^2-a^2}}\mathbf{dx}=\frac{\mathbf{1}}{\mathbf{2a}}\mathbf{\log}\frac{\mathbf{x-a}}{\mathbf{x+a}},\ x>a.$$

If $x<a$, this formula gives an imaginary value for the integral, but we can add the constant $(1/2a)\log(-1)$ to the result and write

$$\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\frac{a-x}{x+a},\ x<a,$$

which is real. This is also the result which we obtain on writing $1/(a^2-x^2)$ as

$$\frac{1}{2a}\left(\frac{1}{a-x}+\frac{1}{a+x}\right),$$

and integrating.

Also, we know by trigonometry the values of the inverse hyperbolic tangent and cotangent in terms of the logarithmic function. Using these results, we can also write :

$$\int \frac{dx}{x^2-a^2}=-\frac{1}{a}\coth^{-1}\frac{x}{a},\ x>a;$$

$$\int \frac{dx}{x^2-a^2}=-\frac{1}{a}\tanh^{-1}\frac{x}{a},\ x<a.$$

Ex. Integrate $\cos mx \cos nx$.

$$\int \cos mx \cos nx\, dx=\tfrac{1}{2}\int\{\cos(m+n)x+\cos(m-n)x\}\,dx$$

$$=\tfrac{1}{2}\left\{\frac{\sin(m+n)x}{m+n}+\frac{\sin(m-n)x}{m-n}\right\}.$$

NOTE. The method of breaking up a given rational algebraic fraction into partial fractions is considered in detail in the next chapter. As regards the simple fractions which occur in the examples of the present chapter, they can be broken up into partial fractions by supposing them to be equal to

$$\frac{A}{x-\alpha}+\frac{B}{x-\beta}, \qquad \ldots \quad (1)$$

where $(x-\alpha)(x-\beta)$ is the denominator of the given fraction,

and comparing the numerator of the given fraction with the numerator in the value of (1), i.e., with $A(x-\beta)+B(x-\alpha)$. We get two equations which determine A and B.

1·37. Reduction formulae. A formula which connects an integral with another in which the integrand is of the same type, but is of lower degree or order or is otherwise easier to integrate, is called a *reduction formula.* Usually the reduction formula has to be used repeatedly to arrive at the integral of the given function. This method of integration is called *integration by successive reduction.*

Imp

Reduction formulae are obtained by one or the other of the preceding methods—very often by the method of integration by parts, and are useful when the integral cannot be otherwise immediately obtained. Various reduction formulae will be obtained in the chapters which follow.

Imp **Ex.** Evaluate $\int \sin^6 x\, dx$. [*Agra*, 1959]

$$\int \sin^n x\, dx = \int \sin^{n-1} x \sin x\, dx$$
$$= \sin^{n-1} x\,(-\cos x) + \int (n-1) \sin^{n-2} x \cos^2 x\, dx,$$

on integration by parts,

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1-\sin^2 x)\, dx$$
$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx.$$

Transposing and dividing by n,

$$\int \sin^n x\, dx = -(1/n) \sin^{n-1} x \cos x + \{(n-1)/n\} \int \sin^{n-2} x\, dx,$$

which is a reduction formula. Applying this successively

$$\int \sin^6 x\, dx = -\tfrac{1}{6} \sin^5 x \cos x + \tfrac{5}{6} \int \sin^4 x\, dx$$

$$= -\tfrac{1}{6} \sin^5 x \cos x + \tfrac{5}{6}\{-\tfrac{1}{4} \sin^3 x \cos x + \tfrac{3}{4} \int \sin^2 x\, dx\}$$

$$= -\tfrac{1}{6} \sin^5 x \cos x - \tfrac{5}{24} \sin^3 x \cos x + \tfrac{5}{8}\{-\tfrac{1}{2} \sin x \cos x + \tfrac{1}{2} \int dx\}$$

$$= -\tfrac{1}{6} \sin^5 x \cos x - \tfrac{5}{24} \sin^3 x \cos x - \tfrac{5}{16} \sin x \cos x + \tfrac{5}{16} x.$$

EXAMPLES

Integrate

1. $\dfrac{1}{(x-1)(x-2)}, \quad \dfrac{2x}{(x-1)(x+3)}, \quad \dfrac{7}{x(x+2)}.$

2. $\dfrac{1}{x^2-2x-3}, \quad \dfrac{x}{x^2+x-6}, \quad \dfrac{1}{x^2-4}.$

3. $\sin x \sin 3x,\ \sin^2 x,\ \sin^3 x.$ 4. $\sin 2x \cos 4x,\ \cos^2 x,\ \cos^3 x.$

1·4. Additional standard forms.

(1) $$\int \operatorname{cosec} x\, dx = \int \frac{dx}{2\sin\frac{1}{2}x\cos\frac{1}{2}x} = \frac{1}{2}\int \frac{\sec^2\frac{1}{2}x\, dx}{\tan\frac{1}{2}x}$$

$$= \int \frac{dt}{t}, \text{ putting } \tan\tfrac{1}{2}x = t \text{ and }$$

$$\tfrac{1}{2}\sec^2\tfrac{1}{2}x\, dx = dt,$$

$$= \log t,$$

i.e., $$\boldsymbol{\int \operatorname{cosec} x\, dx = \log\tan\tfrac{1}{2}x.}$$

(2) $$\int \sec x\, dx = \int \sec(t - \tfrac{1}{2}\pi)\, dt, \quad \text{putting } x = t - \tfrac{1}{2}\pi$$

$$\text{and } dx = dt,$$

$$= \int \operatorname{cosec} t\, dt = \log\tan\tfrac{1}{2}t, \text{ by the above,}$$

i.e., $$\boldsymbol{\int \sec x\, dx = \log\tan(\tfrac{1}{2}x + \tfrac{1}{4}\pi).}$$

ALTERNATIVELY, $$\int \sec x\, dx = \int \frac{\sec x(\tan x + \sec x)\, dx}{\sec x + \tan x}\, dx$$

$$\boldsymbol{= \log(\sec x + \tan x),}$$

by §1·34. It is easy to show by trigonometry that this result is the same as the one obtained above.

(3) $\int \frac{dx}{\sqrt{(x^2+a^2)}} = \int dt = t$, putting $x = a \sinh t$ and $dx = a \cosh t\, dt$, and remembering that

$$1 + \sinh^2 x = \cosh^2 x,$$

i.e., $$\int \frac{\mathbf{dx}}{\sqrt{(\mathbf{x}^2+\mathbf{a}^2)}} = \mathbf{sinh^{-1}(x/a)}.$$

(4) $\int \frac{dx}{\sqrt{(x^2-a^2)}} = \int dt$, putting $x = a \cosh t$ and $dx = a \sinh t\, dt$,

$$= t,$$

i.e., $$\int \frac{\mathbf{dx}}{\sqrt{(\mathbf{x}^2-\mathbf{a}^2)}} = \mathbf{cosh^{-1}(x/a)}.$$

(5) $\int \sqrt{(a^2-x^2)}\, dx = a^2 \int \cos^2 t\, dt$, putting $x = a \sin t$ and $dx = a \cos t\, dt$,

$$= \tfrac{1}{2}a^2 \int (1 + \cos 2t)\, dt$$

$$= \tfrac{1}{2}a^2 t + \tfrac{1}{4}a^2 \sin 2t$$

$$= \tfrac{1}{2}a \sin t \,.\, a \cos t + \tfrac{1}{2}a^2 t,$$

i.e., $$\int \sqrt{(\mathbf{a}^2-\mathbf{x}^2)}\, \mathbf{dx} = \tfrac{1}{2}\mathbf{x}\sqrt{(\mathbf{a}^2-\mathbf{x}^2)} + \tfrac{1}{2}\mathbf{a}^2 \mathbf{sin^{-1}(x/a)}.$$

(6) $\int \sqrt{(x^2+a^2)}\, dx = a^2 \int \cosh^2 t\, dt$, putting $x = a \sinh t$ and $dx = a \cosh t\, dt$,

$$= \tfrac{1}{2}a^2 \int (\cosh 2t + 1)\, dt$$

$$= \tfrac{1}{4}a^2 \sinh 2t + \tfrac{1}{2}a^2 t$$

$$= \tfrac{1}{2}a \sinh t \,.\, a \cosh t + \tfrac{1}{2}a^2 t,$$

i.e., $$\int \sqrt{(\mathbf{x}^2+\mathbf{a}^2)}\, \mathbf{dx} = \tfrac{1}{2}\mathbf{x}\sqrt{(\mathbf{a}^2+\mathbf{x}^2)} + \tfrac{1}{2}\mathbf{a}^2 \mathbf{sinh^{-1}(x/a)}.$$

(7) $\int\sqrt{(x^2-a^2)}\,dx=a^2\int\sinh^2 t\,dt$, putting $x=a\cosh t$ and $dx=a\sinh t\,dt$,

$$=\tfrac{1}{2}a^2\int(\cosh 2t-1)\,dt$$

$$=\tfrac{1}{4}a^2\sinh 2t-\tfrac{1}{2}a^2t,$$

$$=\tfrac{1}{2}a\sinh t\,.\,a\cosh t-\tfrac{1}{2}a^2t,$$

i.e., $\boldsymbol{\int\sqrt{(x^2-a^2)}\,dx=\tfrac{1}{2}x\sqrt{(x^2-a^2)}-\tfrac{1}{2}a^2\cosh^{-1}(x/a)}$.

1·41. Alternative forms and alternative proofs.

As $\sinh^{-1}(x/a)=\log\left\{\frac{x}{a}+\sqrt{\left(\frac{x^2}{a^2}+1\right)}\right\}$,

i.e., $\sinh^{-1}(x/a)=\log\{x+\sqrt{(x^2+a^2)}\}-\log a$,

and $\cosh^{-1}(x/a)=\log\{x+\sqrt{(x^2-a^2)}\}-\log a$,

and we can omit the constant $-\log a$ in stating the result of integration, the results (3), (4), (6) and (7) of the previous article can also be stated as follows :

$$\boldsymbol{\int\frac{dx}{\sqrt{(x^2+a^2)}}=\log\{x+\sqrt{(x^2+a^2)}\},}$$

$$\boldsymbol{\int\frac{dx}{\sqrt{(x^2-a^2)}}=\log\{x+\sqrt{(x^2-a^2)}\},}$$

$$\boldsymbol{\int\sqrt{(x^2+a^2)}\,dx=\tfrac{1}{2}x\sqrt{(x^2+a^2)}+\tfrac{1}{2}a^2\log\{x+\sqrt{(x^2+a^2)}\},}$$

and $$\boldsymbol{\int\sqrt{(x^2-a^2)}\,dx=\tfrac{1}{2}x\sqrt{(x^2-a^2)}-\tfrac{1}{2}a^2\log\{x+\sqrt{(x^2-a^2)}\}.}$$

Moreover, if we wish to avoid using the hyperbolic functions, we can infer the value of $\int dx/\surd(x^2\pm a^2)$ by differentiating $\log\{x+\surd(x^2\pm a^2)\}$.

$$\text{Also} \int\surd(x^2+a^2)\,dx = x\surd(x^2+a^2) - \int\frac{x^2}{\surd(x^2+a^2)}\,dx,$$

on integration by parts,

$$= x\surd(x^2+a^2) - \int\frac{(x^2+a^2)-a^2}{\surd(x^2+a^2)}\,dx$$

$$= x\surd(x^2+a^2) + a^2\int dx/\surd(x^2+a^2) - \int\surd(x^2+a^2)\,dx.$$

Transferring the last term to the left and dividing by 2, we get at once the value of $\int\surd(x^2+a^2)\,dx$. Similarly, we can get the value of $\int\surd(x^2-a^2)\,dx$ by integrating by parts.

1·42. Integrals of $\mathbf{e^{ax}\cos bx}$ and $\mathbf{e^{ax}\sin bx}$.

$$\int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx,$$

on integration by parts,

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left\{\frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx\right\},$$

on integration by parts again.

Transposing the last term and dividing by $1+a^2/b^2$,

$$\int e^{ax}\sin bx\,dx = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx).$$

To simplify the result, put $a=r\cos\theta$ and $b=r\sin\theta$, so that $r=\surd(a^2+b^2)$ and $\theta=\tan^{-1}(b/a)$.

We can always give to r the positive value. Then if a and b are positive, θ is an acute angle, but if a is negative, θ

is an obtuse (and not a negative) angle, because a/r, i.e. $\cos\theta$, is negative. Non-observance of this rule will give a wrong result in numerical cases.

We get $\int e^{ax}\sin bx\,dx = \frac{1}{r}e^{ax}\sin(bx-\theta)$, i.e.,

$$\int \mathbf{e}^{ax}\,\mathbf{\sin bx\,dx} = \frac{\mathbf{1}}{\sqrt{(\mathbf{a}^2+\mathbf{b}^2)}}\,\mathbf{e}^{ax}\,\mathbf{\sin}\left(\mathbf{bx}-\mathbf{\tan}^{-1}\frac{\mathbf{b}}{\mathbf{a}}\right).$$

We can prove similarly that

$$\int \mathbf{e}^{ax}\,\mathbf{\cos bx\,dx} = \frac{\mathbf{1}}{\sqrt{(\mathbf{a}^2+\mathbf{b}^2)}}\,\mathbf{e}^{ax}\,\mathbf{\cos}\left(\mathbf{bx}-\mathbf{\tan}^{-1}\frac{\mathbf{b}}{\mathbf{a}}\right).$$

1·5. Table of standard results to be committed to memory.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1},\; n\neq -1 \qquad \int \tan x\,dx = -\log\cos x$$

$$\int \frac{1}{x}\,dx = \log x \qquad \int \cot x\,dx = \log\sin x$$

$$\int e^x\,dx = e^x \qquad \int \sec x\,dx = \log\tan\left(\tfrac{1}{2}x+\tfrac{1}{4}\pi\right)$$

$$\int a^x\,dx = \frac{a^x}{\log a} \qquad = \log(\sec x+\tan x)$$

$$\int \sin x\,dx = -\cos x \qquad \int \operatorname{cosec} x\,dx = \log\tan\tfrac{1}{2}x$$

$$\int \cos x\,dx = \sin x \qquad \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a} \qquad \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\frac{x-a}{x+a}$$

$$\int \frac{1}{\sqrt{(a^2+x^2)}}\,dx = \sinh^{-1}\frac{x}{a} \text{ or } \log\{x+\sqrt{(x^2+a^2)}\}$$

$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \cosh^{-1}\frac{x}{a} \text{ or } \log\{x+\sqrt{(x^2-a^2)}\}$$

$$\int \sqrt{(a^2-x^2)}\,dx = \tfrac{1}{2}x\sqrt{(a^2-x^2)} + \tfrac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \sqrt{(a^2+x^2)}\,dx = \tfrac{1}{2}x\sqrt{(a^2+x^2)} + \tfrac{1}{2}a^2\sinh^{-1}\left(\frac{x}{a}\right)$$

$$\int \sqrt{(x^2-a^2)}\,dx = \tfrac{1}{2}x\sqrt{(x^2-a^2)} - \tfrac{1}{2}a^2\cosh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{x\sqrt{(x^2-a^2)}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a}$$

$$\int \frac{1}{\sqrt{(2ax-x^2)}}\,dx = \text{vers}^{-1}\frac{x}{a}$$

$$\int e^{ax}\sin bx\,dx = (a^2+b^2)^{-1/2}\,e^{ax}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)$$

$$\int e^{ax}\cos bx\,dx = (a^2+b^2)^{-1/2}\,e^{ax}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)$$

We have also

$\int \sinh x\,dx = \cosh x$	$\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{(n+1)a}$
$\int \cosh x\,dx = \sinh x$	$\int \dfrac{1}{ax+b}\,dx = \dfrac{1}{a}\log(ax+b)$
$\int \text{sech}^2 x\,dx = \tanh x$	$\int e^{ax}\,dx = \dfrac{1}{a}e^{ax}$
$\int \text{cosech}^2 x\,dx = -\coth x,$	$\int \sin(ax+b)\,dx = -\dfrac{1}{a}\cos(ax+b)$
etc.	etc

EXAMPLES

Integrate

1. $2 \operatorname{cosec} x + 3 \sec x$.
2. $5x \sec x^2 + 7$.
3. $\sec(2x+3)$.
4. $\operatorname{cosec}(ax+b)$.
5. $\dfrac{x}{\sqrt{(x^4+4)}}$.
6. $\dfrac{x^2}{\sqrt{(x^6-9)}}$.
7. $\cos x \sqrt{(4-\sin^2 x)}$.
8. $x\sqrt{(x^4+9)}$.
9. $x^2\sqrt{(x^6-1)}$.
10. $\sec x \tan x \sqrt{(\sec^2 x+1)}$.

Imp

1·6. Definite integrals. We define

$$\int_a^b f(x)\,dx$$

to mean $F(b)-F(a)$, where $F(x)$ is an integral of $f(x)$.

$\Big[F(x)\Big]_a^b$ is often used to denote $F(b)-F(a)$.

It should be noted that

$$\Big[F(x)+C\Big]_a^b = \{F(b)+C\} - \{F(a)+C\}$$
$$= F(b)-F(a),$$

and thus the value of $\Big[F(x)+C\Big]_a^b$, and so of $\int_a^b f(x)\,dx$ does not depend upon the value of the arbitrary constant C.

$\int_a^b f(x)\,dx$ is called the *definite integral* of $f(x)$ between the limits a and b.

It is generally read as the "integral from a to b of $f(x)\,dx$". We call b the upper limit and a the lower limit.

It must be remembered that the use of the word "limit" in the present sense is entirely different from the sense in

which we use it when we say that "the limit of $f(x)$ as x tends to c is A."

Ex. 1. $\int_1^2 x^3\,dx = \left[\tfrac{1}{4}x^4\right]_1^2 = \tfrac{1}{4}(16-1) = 15/4.$

Ex. 2. $$\int_0^{\pi/2} \sin^2 x\,dx = \tfrac{1}{2}\int_0^{\pi/2} (1-\cos 2x)\,dx$$
$$= \tfrac{1}{2}\left[x - \tfrac{1}{2}\sin 2x\right]_0^{\pi/2} = \tfrac{1}{4}\pi.$$

Ex. 3. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_1^{\sqrt{3}} = \tfrac{1}{3}\pi - \tfrac{1}{4}\pi = \tfrac{1}{12}\pi.$

NOTE. In the last example, $\tan^{-1}\sqrt{3}$ and $\tan^{-1} 1$ both have an infinite number of values, and so $\tan^{-1}\sqrt{3} - \tan^{-1} 1$ also has an infinite number of values. In all cases like this, where $F(x)$ is a many-valued function, a better definition of the definite integral is required than the one given before. This will be given in Chapter V. Till then, the student should, in determining $F(a)$ and $F(b)$, always take the principal values of the inverse circular functions, viz., those values of $\sin^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, and $\operatorname{cosec}^{-1} x$ which lie between $-\tfrac{1}{2}\pi$ and $\tfrac{1}{2}\pi$ (both values inclusive), and those values of $\cos^{-1} x$, $\sec^{-1} x$, and $\operatorname{vers}^{-1} x$ which lie between 0 and π (both values inclusive).

1·61. Substitution in the case of definite integrals. When the variable in a definite integral is changed, it is usual to change the limits also, in order to avoid the necessity of transforming the result back into the original variable, which is often troublesome.

Now, if we put $\phi(x) = t$, the integral

$$\int_a^b f\{\phi(x)\}\,\phi'(x)\,dx \text{ transforms into } \int f(t)\,dt.$$

Hence

$$\int_a^b f\{\phi(x)\}\,\phi'(x)\,dx \text{ must transform into } \int_{\phi(a)}^{\phi(b)} f(t)\,dt;$$

for, if
$$\int f(t)\,dt = F(t),$$
the latter integral $= \Big[F(t)\Big]_{\phi(a)}^{\phi(b)} = F\{\phi(b)\} - F\{\phi(a)\}$

and the former integral
$$= \Big[F\{\phi(x)\}\Big]_a^b = F\{\phi(b)\} - F\{\phi(a)\},$$
which is the same as before.

We see that *when the variable is changed from x to t, the new limits are the values of t which correspond to the values a and b of x.*

Ex. $\displaystyle\int_2^3 \frac{2x\,dx}{x^2+1} = \int_5^{10} \frac{dt}{t}$, where $t = x^2+1$,
$$= \Big[\log t\Big]_5^{10} = \log 10 - \log 5 = \log_e 2.$$

Here the new lower limit is 5, because when $x=2$, then $t = 2^2+1$, i.e., 5. Similarly, the new upper limit is 10, because $t=10$ when $x=3$.

Important

1·62. Integration by parts in the case of definite integrals. Integration by parts does not present any new difficulties in the case of definite integrals. The following example illustrates the procedure, the validity of which is apparent from the definition.

Ex. Evaluate $\displaystyle\int_0^{\pi/2} x \cos x\,dx$.
$$\int_0^{\pi/2} x\cos x\,dx = \Big[x \,.\, \sin x\Big]_0^{\pi/2} - \int_0^{\pi/2} \sin x\,dx$$
$$= \tfrac{1}{2}\pi + \Big[\cos x\Big]_0^{\pi/2} = \tfrac{1}{2}\pi - 1.$$

Important

1·63. Integrals with infinite limits. We define an integral with one of its limits infinite as follows :

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx,$$

provided the limit is a definite number.

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx,$$

provided the limit is a definite number.

Ex. 1. $\displaystyle\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^3} = \lim_{b\to\infty}\tfrac{1}{2}\left(1-\frac{1}{b^2}\right) = \tfrac{1}{2}.$

Ex. 2. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} 2(\sqrt{b}-1).$

Since $\lim_{b\to\infty}\sqrt{b}$ is not finite, and so is not a definite number, the integral under consideration is meaningless.

Examples

Evaluate

1. $\displaystyle\int_0^1 x^{10}\,dx.$

2. $\displaystyle\int_0^{\pi/2} (\sin x + \cos x)\,dx.$

3. $\displaystyle\int_0^1 \frac{5\,dx}{1+x^2}.$

4. $\displaystyle\int_0^\infty \frac{dx}{2(1+x^2)}.$

5. $\displaystyle\int_{-1}^3 \frac{dx}{2x+3}.$

6. $\displaystyle\int_0^{\pi/4} \cos(x+\tfrac{1}{4}\pi)\,dx.$

7. $\displaystyle\int_0^1 \frac{dx}{\sqrt{(4-x^2)}}.$

8. $\displaystyle\int_0^\infty \frac{dx}{9+x^2}.$

9. $\displaystyle\int_0^a x^2 \sin x^3\,dx.$

10. $\displaystyle\int_1^2 \frac{(1+\log x)^4}{x}\,dx.$

11. $\displaystyle\int_0^1 \frac{5x^3\,dx}{\sqrt{(1-x^8)}}.$

12. $\displaystyle\int_0^1 \frac{(\tan^{-1}x)^2\,dx}{1+x^2}.$

13. $\displaystyle\int_0^{\pi/3} \frac{\cos x\,dx}{3+4\sin x}.$ [*Alig.*,'50]

14. $\displaystyle\int_1^3 \frac{\cos(\log x)\,dx}{x}.$

15. $\int_0^1 x^2 e^{2x}\,dx.$ 16. $\int_0^{\pi/2} \sin^3 x\,dx.$

17. $\int_0^{\pi/2} \cos^4 x\,dx.$ 18. $\int_0^{\pi} \cos^3 x\,dx.$

1·7.* The integral as a sum. Integrals enable us to find the values of the limits of certain sums. The theorem is as follows :

If $$\int f(x)\,dx = F(x) + C,$$

then $\mathbf{lim}_{n\to\infty}\,\mathbf{h[f(a)+f(a+h)+f(a+2h)+\ldots}$
$$\mathbf{+f\{a+(n-1)h\}]=F(b)-F(a),}$$

where $$\mathbf{h=(b-a)/n,}$$

$f(x)$ is a continuous function of x in the domain (a, b) and a and b are fixed finite numbers. Of course, h will tend to zero as n tends to infinity.

We can prove the theorem as follows :

Since $\lim_{h\to 0}\dfrac{F(x+h)-F(x)}{h}=f(x)$,

it follows that $\{F(x+h)-F(x)\}/h=f(x)+\epsilon$,

where $$\epsilon\to 0 \quad \text{as} \quad h\to 0.$$

Giving to x the values $a,\ a+h,\ a+2h,\ \ldots$ in succession, we get

$$\begin{aligned}
F(a+h)-F(a) &= hf(a) && +h\epsilon_1,\\
F(a+2h)-F(a+h) &= hf(a+h) && +h\epsilon_2,\\
F(a+3h)-F(a+2h) &= hf(a+2h) && +h\epsilon_3,\\
\text{etc.} & \quad \text{etc.} &&\\
F(a+nh)-F\{a+(n-1)h\} &= hf\{a+(n-1)h\} && +h\epsilon_n.
\end{aligned}$$

*May be omitted at first reading.

By addition, since $a+nh=b$, we get

$$F(b)-F(a)=h[f(a)+f(a+h)+\dots+f\{a+(n-1)h\}]+h(\epsilon_1+\epsilon_2+\dots+\epsilon_n). \quad (1)$$

Let ϵ_r be the term in $\epsilon_1+\epsilon_2+\dots+\epsilon_n$ which has the largest numerical value. Then

$$h|\epsilon_1+\epsilon_2+\dots+\epsilon_n|\leqslant hn|\epsilon_r|\leqslant(b-a)|\epsilon_r|.$$

As $h\to 0$, $\epsilon_r\to 0$. Hence $(b-a)|\epsilon_r|\to 0$.
Therefore

$$h(\epsilon_1+\epsilon_2+\dots+\epsilon_n)\to 0 \text{ as } h\to 0.$$

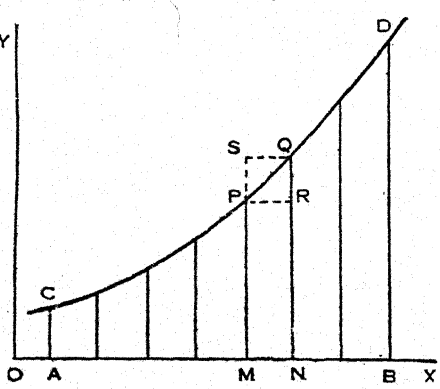
The truth of the theorem to be proved is now obvious from the equation (1) on taking limits as $n\to\infty$.

NOTE. The sum $h[f(a)+f(a+h)+\dots+f\{a+(n-1)h\}]$ is really a function of n, for $h=(b-a)/n$. We have determined above the limit of this as n tends to infinity. But it should be carefully noted that n is not a variable of the type which can take up every numerical value, because n can have only positive integral values. However, the student should not find any difficulty in following the above proof, as he must be familiar with functions of the integral variable n in Algebra, specially in the theory of infinite series.

We have used above the proposition that $\epsilon\to 0$ as $h\to 0$. But in the present case, h can take up only such values as are obtained by giving integral values to n in the expression $(b-a)/n$, whereas we know from Differential Calculus the truth of the above proposition for that case only in which h can take every numerical value between 0 and some positive number. However, it is almost obvious that ϵ will tend to zero also when h takes only the values $(b-a)/n$.

Some other points also require examination, but the discussion will be too subtle for the beginner.

1·8.* Areas. The proposition of §1·7 enables us to find certain areas.



Let DC be the curve $y=f(x)$, CA and DB the ordinates at $x=a$ and $x=b$ respectively.

It is required to find the area $ABDC$, viz., the area bounded by the curve, the axis of x, and the ordinates at $x=a$ and $x=b$.

Divide AB into n parts, each equal to h, so that

$$b-a=nh.$$

Let M, N be the points on OX whose abscissae are $a+rh$ and $a+(r+1)h$ respectively, and let P and Q be the corresponding points on the curve.

We assume, for the sake of convenience,† that $f(x)$ goes on increasing as we go from A to B. We assume also, as an axiom, that the area $MNQP$ lies in magnitude between the areas of the rectangles $MNRP$ and $MNQS$, i.e.,

$$hf\{a+rh\}<\text{area } MNQP<hf\{a+(r+1)h\}.$$

By writing similar inequalities for all the strips into which the area $ABDC$ is divided by the ordinates at $x=a+h, a+2h, \ldots, a+(n-1)h$, and adding, we see that

*May be omitted at first reading.

†This restriction can be easily removed. See below.

$$h[f(a)+f(a+h)+\ldots+f\{a+(n-1)h\}]<\text{area } ABDC$$
$$<h[f(a+h)+\ldots+f(a+nh)]. \quad . \quad . \quad (1)$$

Now the limit, as $n \to \infty$, of the right-hand side

$$=\lim_{n\to\infty} h[f(a)+f(a+h)+f(a+2h)+\ldots$$
$$+f\{a+(n-1)h\}]$$
$$+\lim_{n\to\infty} h[-f(a)+f(a+nh)],$$

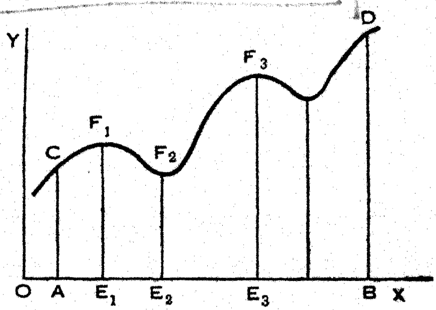
by adding and subtracting $hf(a)$,

$$=F(b)-F(a)+0, \text{by § 1·7, where } \int f(x)\,dx=F(x)+C.$$

The limit, as $n \to \infty$, of the left-hand side also is, by § 1·7, equal to $F(b)-F(a)$.

It follows, therefore, upon taking limits in the inequality (1), that

$$\boxed{\textbf{area ABDC}=\int_a^b f(x)\,dx=\int_a^b \mathbf{y\,dx}.}$$



If $f(x)$ goes on decreasing as we go from A to B, we can show similarly that the proposition is still true.

If, however, $f(x)$ increases in certain parts of the interval AB and decreases in other parts, as in the marginal figure, then the area $ABDC$=the area AE_1F_1C

$$+\text{the area } E_1E_2F_2F_1+\ldots+\text{the area } E_nBDF_n,$$

where E_1F_1, E_2F_2, ..., E_nF_n are the maximum or minimum ordinates of the curve (n being finite). If the abscissae of E_1, E_2, ..., E_n are c_1, c_2, ..., c_n, it follows from the above that the area $ABDC$

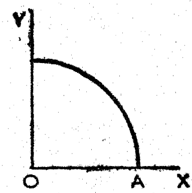
$$=\{F(c_1)-F(a)\}+\{F(c_2)-F(c_1)\}+\ldots+\{F(b)-F(c_n)\}$$
$$=F(b)-F(a).$$

Hence the proposition is still true, provided that the curve has a finite number of turning points.

The area bounded by the curve CD, the ordinates at C and D, and the x-axis is often called *the area under the curve CD*.

Ex. Find the area of a quadrant of the circle

$$x^2+y^2=1.$$



The quadrant of the circle may be regarded as bounded by the curve $y=\sqrt{(1-x^2)}$, the x-axis and the ordinates at $x=0$ and $x=1$.

Hence the required area $=\int_0^1 \sqrt{(1-x^2)}\,dx$

$$=\left[\tfrac{1}{2}\sin^{-1}x+\tfrac{1}{2}x\sqrt{(1-x^2)}\right]_0^1$$
$$=\tfrac{1}{2}(\sin^{-1}1-0)=\pi/4.$$

Note. Very often, in the application of the integral calculus to physics and to other subjects, an argument similar to the following is used:

To find the area $ABDC$ (see figure on page 32), divide it into n strips each of breadth dx. Then the area of any one strip $MNQP$ is approximately $PM\,.\,dx$, i.e., $f(x)\,dx$.

Hence the total area, viz., the area $ABDC=\int_a^b f(x)\,dx$.

This procedure must be regarded as merely a rough abbreviation of the procedure we have adopted; otherwise it has no meaning.

Examples on Chapter I

Integrate with respect to x:

1. $(\tan^{-1}x)/(1+x^2)$.
2. $1/(1+x^2)\tan^{-1}x$.
3. $\dfrac{\cot(\log x)}{x}$.
4. $\dfrac{e^{m\sin^{-1}x}}{\sqrt{(1-x^2)}}$.
5. $\dfrac{x}{(x^2+3)^2}$.
6. $\dfrac{\cot x}{\log\sin x}$. [*Aligarh*, '60]

7. $\dfrac{x}{1+x^4}$. [*Alig.*, '60]

8. $\dfrac{1}{\sin 2x}$.

9. $\dfrac{a}{b+ce^x}$. [*Alld.*, 1953]

10. $\dfrac{1}{e^x-1}$. [*Gorakhpur*, '60]

11. $\sec 4x$.

12. $x \sin x$. [*Ban. Geo.*, '57]

13. $x \tan^{-1} x$. [*Alig.*, '60]

14. $x \sec^2 x$. [*Agra*, 1951]

15. $\sin x \sinh x$.

Ask 16. $x \sin^{-1} x$. [*Rajasthan*, '62]

Imp 17. $\dfrac{x \tan^{-1} x}{(1+x^2)^{3/2}}$. [*U.P.For.*, '60]

18. Ask $\dfrac{x \sin^{-1} x}{\sqrt{(1-x^2)}}$. [*Gorakh.*, '60]

19. $x^2 \log x$. [*Sagar*, '60]

20. $x^{-n} \log x$. [*Nagpur*, '56]

21. $x^2 e^x$.

22. $x^2 e^{3x}$.

23. $x^2 (\log x)^2$. [*Utkal*, '50]

24. $x^3 \sin ax$.

Ask 25. $x^3 e^{x^2}$. [*Banaras, Geophysics*, 1956]

26. $x^2(\tan^{-1} x)/(1+x^2)$. [*Delhi*, 1960]

Imp 27. $\dfrac{x+\sin x}{1+\cos x}$. [*Banaras*, '60]

28. $\dfrac{3x}{x^2-x-2}$.

Ask 29. $\dfrac{x-1}{(x-3)(x-2)}$. [*Roor.*, '62]

30. $\dfrac{x}{(x^2-a^2)(x^2-b^2)}$. [*Nag.*, '53]

Evaluate

31. $\displaystyle\int_0^1 \frac{1-x}{1+x}\, dx$. [*Utkal*, '50]

32. $\displaystyle\int_1^2 \frac{dx}{x(1+x^4)}$. [*Nag.*, '56]

33. $\displaystyle\int \frac{\cos 2x}{\cos x}\, dx$.

Ask 34. $\displaystyle\int \frac{dx}{\sinh x}$.

Ask 35. $\displaystyle\int \sin 2x \cos 3x\, dx$.

36. $\displaystyle\int_0^{\pi/2} e^x (\sin x + \cos x)\, dx$.

37. $\int e^{3x} \sin 4x\, dx$. [*Panjab*, 1954]

38. $\int \sqrt{(1+\sin x)}\, dx$. [*Allahabad*, 1962]

[Hint : Notice that $\sqrt{(1+\sin x)} = \cos \frac{1}{2}x + \sin \frac{1}{2}x$
$= \sqrt{2} \sin(\frac{1}{2}x + \frac{1}{4}\pi)$.]

39. Integrate and evaluate

$$\text{(i)} \int_0^{\pi/2} \cos^2 x\, dx, \quad \text{(ii)} \int_0^{\pi/4} \tan^2 x\, dx, \quad \text{(iii)} \int_0^1 xe^x\, dx.$$

Imp

40. Show that

$$\int_a^b \frac{\log x}{x}\, dx = \tfrac{1}{2} \log\left(\frac{b}{a}\right) \log(ab).$$

Most Imp

41. Evaluate the following :

$$\int \frac{xe^x}{(x+1)^2}\, dx. \qquad \text{[\textit{Lucknow}, 1962]}$$

[Hint : Integrate by parts, taking xe^x as the first function].

42. Integrate $x^3/(x^2+1)^3$, (*a*) by the substitution $x = \tan\theta$, (*b*) by the substitution $u = x^2+1$, and show that the results you obtain by the two methods are in accordance.

V.V.V. Imp

43. Prove that $\displaystyle\int u \frac{d^2v}{dx^2}\, dx = u\frac{dv}{dx} - v\frac{du}{dx} + \int v \frac{d^2u}{dx^2}\, dx.$

[*Lucknow*, 1944]

44. Prove that if the integral of a function is known, the integral of the inverse function can be deduced. Illustrate graphically. Find

$$\int \tan^{-1} x\, dx.$$

[Put the inverse function, say $f^{-1}(x)$, equal to t, so that $x = f(t)$, $dx = f'(t)\, dt$, and integrate the resulting function by parts. Graphical representation is very similar to that in § 5·5 (iii).]

45. Find a reduction formula for $\int x^n \sin ax\, dx$.

46. Find the area, between the curve, the x-axis and the ordinates at $x=1$ and $x=2$, of

$$\text{(i) } y = x^2, \quad \text{(ii) } y = e^x.$$

47. Find the area between the x-axis and the curve $y = \sin x$ from $x=0$ to $x=\pi$.

48. Show that the area between the curve $y = ce^x$, the x-axis and any two ordinates is proportional to the difference between the ordinates.

CHAPTER II
INTEGRATION OF RATIONAL FRACTIONS

2·1. Partial fractions. The fraction

$$\frac{a_0x^m+a_1x^{m-1}+\ldots+a_m}{b_0x^n+b_1x^{n-1}+\ldots+b_n},$$

in which a_0, a_1, ..., b_0, b_1, ... are constants and m and n positive integers, is called a rational algebraic fraction. Such fractions can always be integrated by breaking them up into the sum of an integral part and a number of *partial fractions*, i.e., fractions in which the denominators are linear or quadratic functions of x and the numerators are of a lower degree than the denominators.

Let the numerator of the given fraction be written as $F(x)$ and the denominator as $\phi(x)$. If $F(x)$ is not of a lower degree than $\phi(x)$, divide $F(x)$ by $\phi(x)$ till the remainder, say $f(x)$, is of a lower degree than $\phi(x)$. If the quotient be $Q(x)$, then the given fraction is equal to

$$Q(x)+\frac{f(x)}{\phi(x)}.$$

$Q(x)$, being the sum of a number of terms like C_rx^r, can be integrated at once. We proceed to consider how $f(x)/\phi(x)$ can be broken up into partial fractions. Henceforward it will be assumed that the numerator of the fraction under consideration is of a lower degree than the denominator.

First resolve the denominator into its real prime factors. These factors will be either linear

or quadratic, and some of the factors may occur more than once. We know from algebra that $f(x)/\phi(x)$ can be resolved in one and only one way into a sum of partial fractions, which are of the following types:

(i) To every non-repeated linear factor $(x-a)$ in the denominator corresponds a partial fraction of the form $A/(x-a)$.

(ii) To every linear factor repeated r times, i.e., to every factor of the type $(x-b)^r$, correspond r partial fractions of the form

$$\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\frac{B_3}{(x-b)^3}+\ldots+\frac{B_r}{(x-b)^r}.$$

(iii) To every non-repeated quadratic factor x^2+px+q corresponds a partial fraction of the form

$$(Cx+D)/(x^2+px+q).$$

(iv) To every quadratic factor repeated s times, i.e., to every factor of the type $(x^2+kx+l)^s$, correspond s partial fractions of the form

$$\frac{E_1x+F_1}{x^2+kx+l}+\frac{E_2x+F_2}{(x^2+kx+l)^2}+\ldots+\frac{E_sx+F_s}{(x^2+kx+l)^s}.$$

The next step is to determine the coefficients $A, B, C, \ldots$. Methods for doing this will be given in the articles which follow.

2·2. Non-repeated linear factors only in the denominator. Suppose there is a non-repeated linear factor $(x-a)$ in the denominator $\phi(x)$, and let $\phi(x)=(x-a)\psi(x)$. Then $f(x)/\phi(x)$,

i.e., $$\frac{f(x)}{(x-a)\psi(x)}=\frac{A}{x-a}+\text{partial fractions not containing } x-a \text{ in the denominator.}$$

Multiplying both sides by $x-a$, we have

$f(x)/\psi(x)=A+(x-a)\times$partial fractions not containing $x-a$ in the denominator.

Putting $x=a$ in this identity, we get

$$f(a)/\psi(a)=A,$$

i.e., $$\frac{A}{x-a}=\frac{f(a)}{(x-a)\psi(a)}.$$

Since the right-hand side can be obtained from the given fraction by putting in it $x=a$, everywhere except in the factor $x-a$, we get the rule: *to obtain the partial fraction corresponding to the factor $x-a$ in the denominator, put $x=a$ everywhere in the given fraction except in the factor $x-a$ itself.*

All the partial fractions can be written down with the help of the above rule when the denominator of the given rational fraction contains only non-repeated linear factors.

We note that $\phi(x)=(x-a)\psi(x)$.

Hence $\phi'(x)=(x-a)\psi'(x)+\psi(x)$.

Therefore $\phi'(a)=\psi(a)$.

Hence the value of A can also be written as

$$f(a)/\phi'(a).$$

Ex. Integrate $(x^2+x+2)/(x-2)(x-1)$. [*Baroda*, '60]

Since the numerator is not of a lower degree than the denominator, we first divide out. We thus find that the given fraction is equal to

$$1+\frac{4x}{(x-2)(x-1)}.$$

Now, by the above rule,

$$\frac{4x}{(x-2)(x-1)}=\frac{4\times2}{(x-2)(2-1)}+\frac{4\times1}{(1-2)(x-1)},$$

i.e.,
$$\frac{x^2+x+2}{(x-2)(x-1)}=1+4\left(\frac{2}{x-2}-\frac{1}{x-1}\right).$$

Hence
$$\int\frac{x^2+x+2}{(x-2)(x-1)}\,dx=\int\left\{1+4\left(\frac{2}{x-2}-\frac{1}{x-1}\right)\right\}dx$$
$$=x+4\{2\log(x-2)-\log(x-1)\}$$
$$=x+4\log\{(x-2)^2/(x-1)\}.$$

EXAMPLES

Integrate

1. $(x^2+1)/(x^2-1)$. 2. $x^3/(x-1)(x+2)(2x+3)$.
3. $x^2/(x+1)(x-2)(x+3)$. [*Banaras, Geophysics*, 1957]
4. $x^2/(x+1)(x+2)(x+3)$. [*Jammu*, 1954]
5. $x^2/(x-1)(3x-1)(3x-2)$. [*Patna*, 1937]
6. $x^5/(x^3-2x^2-5x+6)$.
7. $x/(x-a)(x-b)(x-c)$.
8. $(x-a)(x-b)(x-c)/(x-\alpha)(x-\beta)(x-\gamma)$. [*Baroda*, '59]

2·3. Repeated factors. Next we consider the case in which the denominator of the given fraction has linear factors, some of which occur more than once. Suppose the given fraction is $f(x)/\phi(x)$, where $\phi(x)=(x-a)^r\psi(x)$. To find the partial fractions corresponding to $(x-a)^r$, put $x-a=y$ in the given fraction. Then $f(x)/\phi(x)$, i.e.,

$$\frac{f(x)}{(x-a)^r\psi(x)}=\frac{f(a+y)}{y^r\psi(a+y)}$$
$$=\frac{1}{y^r}\frac{A_0+A_1y+A_2y^2+\dots}{B_0+B_1y+B_2y^2+\dots},\text{ say,}$$

when the numerator and denominator are arranged in ascending powers of y. Now divide $A_0+A_1y+\ldots$ by $B_0+B_1y+\ldots$ and continue the process till y^r becomes a common factor of the remainder. Suppose the quotient is $C_0+C_1y+\ldots+C_{r-1}y^{r-1}$ and the remainder is $y^r(D_0+D_1y+\ldots)$, so that

$$\frac{A_0+A_1y+A_2y^2+\ldots}{B_0+B_1y+B_2y^2+\ldots}=C_0+C_1y+\ldots+C_{r-1}y^{r-1}$$
$$+\frac{y^r(D_0+D_1y+\ldots)}{B_0+B_1y+\ldots}.$$

Then $$\frac{f(x)}{\phi(x)}=\frac{C_0}{y^r}+\frac{C_1}{y^{r-1}}+\ldots+\frac{C_{r-1}}{y}+\frac{D_0+D_1y+\ldots}{B_0+B_1y+\ldots}$$
$$=\frac{C_0}{(x-a)^r}+\frac{C_1}{(x-a)^{r-1}}+\ldots+\frac{C_{r-1}}{x-a}$$
$$+\frac{D_0+D_1(x-a)+\ldots}{\psi(x)}.$$

Thus the partial fractions which have $(x-a)^r$, $(x-a)^{r-1}$, etc. in their denominators have been determined. The fraction

$$\{D_0+D_1(x-a)+\ldots\}/\psi(x)$$

can now be further broken up into partial fractions by the method of the present article or of the last article, as the case may be.

Ex. Integrate $x^3/(x+1)^4(x+2)(x-1)$.

Put $x+1=y$. Then the given fraction is equal to

$$\frac{1}{y^4}\,\frac{(-1+y)^3}{(1+y)(-2+y)}.$$

Divide the numerator of the fraction which is the factor of $1/y^4$ by its denominator, as shown below.

$$
\begin{array}{l}
-2-y+y^2)\,-1+3y-3y^2+y^3\left(\tfrac{1}{2}-\tfrac{7}{4}y+\tfrac{21}{8}y^2-\tfrac{43}{16}y^3\right. \\
\qquad\qquad\quad \underline{-1-\tfrac{1}{2}y+\tfrac{1}{2}y^2} \\
\qquad\qquad\qquad\quad \tfrac{7}{2}y-\tfrac{7}{2}y^2+\ y^3 \\
\qquad\qquad\qquad\quad \underline{\tfrac{7}{2}y+\tfrac{7}{4}y^2-\tfrac{7}{4}y^3} \\
\qquad\qquad\qquad\qquad\quad -\tfrac{21}{4}y^2+\tfrac{11}{4}y^3 \\
\qquad\qquad\qquad\qquad\quad \underline{-\tfrac{21}{4}y^2-\tfrac{21}{8}y^3+\tfrac{21}{8}y^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad \tfrac{43}{8}y^3-\tfrac{21}{8}y^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{\tfrac{43}{8}y^3+\tfrac{43}{16}y^4-\tfrac{43}{16}y^5} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\tfrac{85}{16}y^4+\tfrac{43}{16}y^5.
\end{array}
$$

We see that the given fraction

$$
\begin{aligned}
&=\frac{1}{2y^4}-\frac{7}{4y^3}+\frac{21}{8y^2}-\frac{43}{16y}+\frac{-\frac{85}{16}+\frac{43}{16}y}{-2-y+y^2} \\
&=\frac{1}{2(x+1)^4}-\frac{7}{4(x+1)^3}+\frac{21}{8(x+1)^2}-\frac{43}{16(x+1)} \\
&\qquad\qquad+\frac{-85+43(x+1)}{16(x+2)(x-1)}.
\end{aligned}
$$

Now the last term

$$
\begin{aligned}
=\frac{43x-42}{16(x+2)(x-1)}&=\frac{-128}{16(x+2)(-2-1)}+\frac{1}{16(1+2)(x-1)} \\
&=\frac{8}{3(x+2)}+\frac{1}{48(x-1)}.
\end{aligned}
$$

Hence the integral of the given fraction

$$
\begin{aligned}
=-\frac{1}{6(x+1)^3}&+\frac{7}{8(x+1)^2}-\frac{21}{8(x+1)}-\frac{43}{16}\log(x+1) \\
&+\tfrac{8}{3}\log(x+2)+\tfrac{1}{48}\log(x-1).
\end{aligned}
$$

Examples

Integrate

1. $x/(x-1)^2(x+2)$. [*Banaras, Geophysics*, 1956]
2. $(x^2+1)/(x+1)^3(x-2)$.
3. $(x+1)/x^4(x-1)$.
4. $1/x(x+1)^2$.
5. $\dfrac{x^3+2}{(x-1)(x-2)^3}$. [*Gor.*, '60]
6. $\dfrac{x+1}{(x-1)^2(x+2)^2}$.

2·4. General case. If there are quadratic factors in the denominator which cannot be resolved into real linear factors, a good plan is as follows :

Equate the fraction to a sum of partial fractions of the correct form in accordance with § 2·1. Find first the constants which can be determined by the methods of the previous articles. To find the remaining constants, multiply both sides by the denominator of the given fraction and equate the coefficients of like powers of x in the resulting identity. Choose the simplest of the equations thus obtained which will give the values of the unknown constants, and solve them.

Ex. Break up $1/(x^3+1)$ into its partial fractions.

$$\frac{1}{x^3+1}=\frac{1}{(x+1)(x^2-x+1)}=\frac{1}{(x+1)(1+1+1)}+\frac{Ax+B}{x^2-x+1}$$

$$=\frac{\frac{1}{3}(x^2-x+1)+(Ax+B)(x+1)}{(x+1)(x^2-x+1)}.$$

Therefore $\frac{1}{3}+A=0$, $-\frac{1}{3}+A+B=0$, and $\frac{1}{3}+B=1$.

The first and the last equations give

$$A=-\tfrac{1}{3},\quad B=\tfrac{2}{3}.$$

Hence $$\frac{1}{x^3+1}=\frac{1}{3(x+1)}-\frac{x-2}{3(x^2-x+1)}.$$

2·41. Labour-saving devices. After equating the fraction to the sum of partial fractions of the correct form, and finding the constants which can be determined by the methods of articles 2·2 and 2·3, we can get a sufficient number of simple equations for determining the remaining coefficients by giving to x a number of convenient special

values, or by allowing x to tend to infinity after multiplying the equation throughout by a suitable power of x.

Ex. Break up $1/(x^3+1)$ into partial fractions.

As in the Example of § 2·4,

$$\frac{1}{x^3+1}=\frac{1}{3(x+1)}+\frac{Ax+B}{x^2-x+1}. \qquad . \quad . \quad (1)$$

Multiplying throughout by x and letting $x\to\infty$, we get

$$0=\tfrac{1}{3}+A. \qquad . \quad . \quad . \quad . \quad (2)$$

Again, putting $x=0$ in (1), we get $1=\frac{1}{3}+B$. . . (3)

Equations (2) and (3) determine A and B.

NOTE. Had there been more constants on the right-hand side of (1), we would have required more equations like (2) and (3). These could have been written down by giving other convenient values to x.

Ex. Resolve $x^2/(x^4+x^2+1)$ into partial fractions.

As $x^4+x^2+1=(x^2+x+1)(x^2-x+1)$, assume that

$$\frac{x^2}{x^4+x^2+1}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}.$$

Multiplying by x and letting $x\to\infty$, we get $0=A+C$.

Again, giving to x successively the values 0, 1, -1, we get

$$0=B+D,\ \tfrac{1}{3}=\tfrac{1}{3}(A+B)+C+D,$$
$$\tfrac{1}{3}=-A+B+\tfrac{1}{3}(-C+D).$$

These four equations give $A=-\frac{1}{2}$, $B=0$, $C=\frac{1}{2}$, $D=0$, and thus the partial fractions are completely determined.

To compare the present method with that of § 2·4, compare this solution with the one given in Ex. 2 of § 2·6.

2·5. Integration of $1/(ax^2+bx+c)$. To integrate $1/(ax^2+bx+c)$, we must put the denominator in the form $a\{(x+\alpha)^2\pm\beta^2\}$. Thus

$$\int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{dx}{x^2+(b/a)x+(b/2a)^2+c/a-(b/2a)^2},$$

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which we may arrange as

$$\frac{1}{a}\int\frac{dx}{\{x+b/2a\}^2+\{c/a-(b/2a)^2\}},$$

or

$$\frac{1}{a}\int\frac{dx}{\{x+b/2a\}^2-\{(b/2a)^2-c/a)\}}.$$

If $(c/a)-(b/2a)^2$ is positive, i.e., if $b^2<4ac$, we take the first form and from the formula

$$\int dx/(x^2+a^2)=(1/a)\tan^{-1}(x/a)$$

obtain

$$\int\frac{dx}{ax^2+bx+c}=\frac{2}{\sqrt{(4ac-b^2)}}\tan^{-1}\frac{2ax+b}{\sqrt{(4ac-b^2)}}.$$

If, however, $(c/a)-(b/2a)^2$ is negative, i.e., if $b^2>4ac$, we take the second form and from the formula

$$\int dx/(x^2-a^2)=(1/2a)\log\{(x-a)/(x+a)\}$$

obtain

$$\int\frac{dx}{ax^2+bx+c}=\frac{1}{\sqrt{(b^2-4ac)}}\log\frac{2ax+b-\sqrt{(b^2-4ac)}}{2ax+b+\sqrt{(b^2-4ac)}}.$$

These two values of the integral differ only by a constant, but in numerical examples the real form should be chosen. Moreover, in the case when $b^2>4ac$, the denominator can be resolved into real linear factors, and we can integrate $1/(ax^2+bx+c)$ by breaking it up into its partial fractions if we so desire.

Ex. 1. $$\int\frac{dx}{2x^2+x+1}=\tfrac{1}{2}\int\frac{dx}{x^2+\frac{1}{2}x+\frac{1}{16}+\frac{1}{2}-\frac{1}{16}}$$

$$=\tfrac{1}{2}\int\frac{dx}{(x+\frac{1}{4})^2+\frac{7}{16}}$$

$$=\tfrac{1}{2}\cdot\frac{4}{\sqrt{7}}\tan^{-1}\frac{x+\frac{1}{4}}{(\sqrt{7})/4}$$

$$=\frac{2}{\sqrt{7}}\tan^{-1}\frac{4x+1}{\sqrt{7}}.$$

Ex. 2.
$$\int \frac{dx}{2x^2+x-1} = \tfrac{1}{2}\int \frac{dx}{x^2+\frac{1}{2}x+\frac{1}{16}-\frac{1}{2}-\frac{1}{16}}$$
$$= \tfrac{1}{2}\int \frac{dx}{(x+\frac{1}{4})^2-\frac{9}{16}} = \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{4}{3}\log\frac{x+\frac{1}{4}-\frac{3}{4}}{x+\frac{1}{4}+\frac{3}{4}},$$

or
$$\tfrac{1}{3}\log\frac{2x-1}{x+1},$$

on omitting a constant.

2·6. Integration of $(px+q)/(ax^2+bx+c)$. The integration of $(px+q)/(ax^2+bx+c)$ is effected by breaking it up into two fractions such that in one the numerator is the differential coefficient of the denominator, and in the other the numerator is merely a constant (the denominator in both cases being ax^2+bx+c). Thus

$$\int \frac{(px+q)\,dx}{ax^2+bx+c} = \frac{p}{2a}\int \frac{2ax+b}{ax^2+bx+c}\,dx + \int \frac{q-(bp/2a)}{ax^2+bx+c}\,dx$$
$$= \frac{p}{2a}\log(ax^2+bx+c) + \int \frac{q-(bp/2a)}{ax^2+bx+c}\,dx.$$

The integral on the right can be easily evaluated by §2·5.

NOTE. It will be futile to break up the given integrand into two others both of which contain x in the numerator. The student will have no difficulty in breaking up the given integrand in the correct way if he remembers that in the first part (viz., in the part in which the numerator is the differential coefficient of the denominator) the coefficient of x in the numerator is made equal to the coefficient of x in the numerator of the original integrand by *multiplying and dividing* by suitably chosen numbers, and the numerator of the second part is the constant in the numerator of the given integrand *increased or diminished* by a number so chosen that the sum of the two parts may be equal to the original fraction.

Ex. 1. Integrate $(3x+1)/(2x^2+x+1)$. [*Baroda*, '60]

$$\int \frac{3x+1}{2x^2+x+1}\,dx = \tfrac{3}{4}\int \frac{4x+1}{2x^2+x+1}\,dx + \int \frac{1-\frac{3}{4}}{2x^2+x+1}\,dx$$

$$= \tfrac{3}{4}\log(2x^2+x+1) + \frac{1}{2\sqrt{7}}\tan^{-1}\frac{4x+1}{\sqrt{7}}, \text{ by § 2·5, Ex. 1.}$$

Ex. 2. Integrate $x^2/(x^4+x^2+1)$. [*Nagpur*, 1959]

Since $x^4+x^2+1=(x^2+1)^2-x^2=(x^2+x+1)(x^2-x+1)$,

assume that $\dfrac{x^2}{x^4+x^2+1} = \dfrac{Ax+B}{x^2+x+1} + \dfrac{Cx+D}{x^2-x+1}$.

Then $x^2 \equiv (Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)$.

Hence $A+C=0,\ -A+B+C+D=1,$
$A-B+C+D=0,\ B+D=0.$

Therefore $A=-\frac{1}{2},\ C=\frac{1}{2},\ B=0,\ D=0$.

Thus $$\int \frac{x^2\,dx}{x^4+x^2+1} = \tfrac{1}{2}\int \frac{x\,dx}{x^2-x+1} - \tfrac{1}{2}\int \frac{x\,dx}{x^2+x+1}$$

$$= \tfrac{1}{4}\int \frac{(2x-1)\,dx}{x^2-x+1} + \tfrac{1}{4}\int \frac{dx}{x^2-x+1} - \tfrac{1}{4}\int \frac{(2x+1)\,dx}{x^2+x+1} + \tfrac{1}{4}\int \frac{dx}{x^2+x+1}$$

$$= \tfrac{1}{4}\log(x^2-x+1) + \tfrac{1}{4}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} - \tfrac{1}{4}\log(x^2+x+1)$$
$$+ \tfrac{1}{4}\int dx/\{(x+\tfrac{1}{2})^2+\tfrac{3}{4}\}$$

$$= \tfrac{1}{4}\log\frac{x^2-x+1}{x^2+x+1} + \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x-\frac{1}{2}}{\sqrt{3}/2} + \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\sqrt{3}/2}$$

$$= \tfrac{1}{4}\log\frac{x^2-x+1}{x^2+x+1} + \frac{1}{2\sqrt{3}}\tan^{-1}\frac{\sqrt{3}x}{1-x^2}.$$

Examples

Integrate

1. $1/(2x^2+x+3)$. [*Jammu*, 1956]
2. $1/(x^2+2x+5)$.
3. $(3x+1)/(2x^2-2x+3)$. [*Sagar*, 1954]
4. $(5x-2)/(1+2x+3x^2)$. [*Nagpur*, 1956]

Evaluate

5. $\displaystyle\int_0^1 \frac{dx}{1-x+x^2}$. [*Ban. Geoph.*, '61] 6. $\displaystyle\int_0^1 \frac{(x-3)\,dx}{x^2+2x-4}$

7. $\int_0^1 \frac{x^3\,dx}{(x^2+1)(x^2+7x+12)}$. [*Allahabad*, 1955]

8. $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$. [*P.S.C., U.P., Forest*, 1955]

Integrate

9. $1/(x^3-1)$. [*Roor.*, '62] 10. $1/(x+1)^2(x^2+1)$. [*Mad.*, '50]

11. $(x^2+1)/(x^3+1)$. 12. $(1-3x)/(1+x^2)(1+x)$.

2·7. Integration of $\mathbf{1/(x^2+k)^n}$. We can integrate $1/(x^2+k)^n$ by the method of successive reduction. To obtain a reduction formula, we integrate $1/(x^2+k)^{n-1}$ by parts, taking unity as one of the factors. Thus

$$\int \frac{dx}{(x^2+k)^{n-1}} = \frac{x}{(x^2+k)^{n-1}} + \int \frac{2(n-1)x^2}{(x^2+k)^n}\,dx$$

$$= \frac{x}{(x^2+k)^{n-1}} + 2(n-1)\int \frac{x^2+k-k}{(x^2+k)^n}\,dx$$

$$= \frac{x}{(x^2+k)^{n-1}} + 2(n-1)\int \frac{dx}{(x^2+k)^{n-1}}$$

$$-2(n-1)k\int \frac{dx}{(x^2+k)^n}.$$

Dividing by $2(n-1)k$ and transposing, we get

$$\int \frac{dx}{(x^2+k)^n} = \frac{x}{(2n-2)k(x^2+k)^{n-1}}$$

$$+\frac{2n-3}{(2n-2)k}\int \frac{dx}{(x^2+k)^{n-1}},$$

which is the required reduction formula.

Ex. Integrate $1/(x^2+3)^3$. By the reduction formula

$$\int \frac{dx}{(x^2+3)^3} = \frac{x}{12(x^2+3)^2} + \frac{3}{12}\int \frac{dx}{(x^2+3)^2}$$

$$=\frac{x}{12(x^2+3)^2}+\tfrac{1}{4}\left\{\frac{x}{6(x^2+3)}+\tfrac{1}{6}\int\frac{dx}{x^2+3}\right\}$$

$$=\frac{x}{12(x^2+3)^2}+\frac{x}{24(x^2+3)}+\frac{1}{24\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}.$$

2·8. Integration of $(\mathbf{px+q})/(\mathbf{ax^2+bx+c})^n$. The integration of $(px+q)/(ax^2+bx+c)^n$ is effected by breaking it up into two parts in one of which the numerator is the differential coefficient of ax^2+bx+c and in the other there is no x in the numerator. Thus

$$\int\frac{(px+q)\,dx}{(ax^2+bx+c)^n}=\frac{p}{2a}\int\frac{(2ax+b)\,dx}{(ax^2+bx+c)^n}+\int\frac{(q-bp/2a)\,dx}{(ax^2+bx+c)^n}$$

$$=\frac{-p}{2a(n-1)(ax^2+bx+c)^{n-1}}$$

$$+\left(q-\frac{bp}{2a}\right)\frac{1}{a^n}\int\frac{dx}{\{x^2+(b/a)x+c/a\}^n}.$$

But

$$\int\frac{dx}{\{x^2+(b/a)x+c/a\}^n}=\int\frac{dx}{\{(x+b/2a)^2+c/a-b^2/4a^2\}^n}.$$

Putting $x+b/2a=t$ and writing k for $c/a-b^2/4a^2$, the integral takes the form $\int dt/(t^2+k)^n$ and so can be evaluated, as in the last article, by successive reduction. Thus $\int\{(px+q)/(ax^2+bx+c)^n\}dx$ can be completely evaluated.

Ex. Integrate $(x+2)/(2x^2+4x+3)^2$. [*Allahabad*, 1950]

$$\int\frac{(x+2)\,dx}{(2x^2+4x+3)^2}=\tfrac{1}{4}\int\frac{(4x+4)\,dx}{(2x^2+4x+3)^2}+\int\frac{(2-1)\,dx}{4(x^2+2x+\frac{3}{2})^2}$$

$$=-\frac{1}{4(2x^2+4x+3)}+\tfrac{1}{4}\int\frac{dx}{\{(x+1)^2+\frac{1}{2}\}^2}$$

$$=-\frac{1}{4(2x^2+4x+3)}$$

$$+\tfrac{1}{4}\left[\frac{x+1}{(x+1)^2+\frac{1}{2}}+\sqrt{2}\tan^{-1}\{(\sqrt{2})(x+1)\}\right], \text{ by § 2·7.}$$

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2·9. Special cases. (i) In certain cases a substitution materially shortens the work. This is specially so if some power of x, say x^{p-1}, is a factor of the numerator, and the rest of the fraction is a rational function of x^p.

(ii) In fractions in which there is no odd power of x and in which the denominator can be broken up into factors of the form $x^2 \pm a^2$, it is not necessary to resolve the denominator into linear factors. The partial fraction corresponding to each factor x^2+a^2 or x^2-a^2 should be obtained by regarding x^2 as the variable.

(iii) If the denominator of the integral is of the form $x^4+kx^2+c^2$ and cannot be resolved into real factors of the form x^2+a, and the numerator is of the form ax^2+b, the substitutions $x+c/x=t$ and $x-c/x=u$ will simplify the work. For, on dividing the numerator and denominator by x^2, the new denominator will be x^2+k+c^2/x^2, which can be written as $(x+c/x)^2+k-2c$ and also as $(x-c/x)^2+k+2c$. Hence we can assume that

$$\frac{ax^2+b}{x^4+kx^2+c^2}=\frac{A(1-c/x^2)}{(x+c/x)^2+k-2c}+\frac{B(1+c/x^2)}{(x-c/x)^2+k+2c},$$

and easily find A and B by comparing the numerators. The terms on the right can now be integrated by putting $x+1/x=t$ and $x-1/x=u$ in them respectively.

(iv) Sometimes it is more convenient to break up the denominator completely into linear factors, athough this may introduce imaginary numbers. After resolution into partial fractions, or after integration, the pairs of terms corresponding to conjugate roots can be combined and reduced to a real form by the help of De Moivre's theorem.

(v) Very often expressions involving x^2+a^2 can be integrated more conveniently by the substitution $x=a\tan\theta$. Examples of this method will be given in Chapter IV.

Ex. 1. Integrate $1/x(x^4-1)$. [*Baroda*, 1960]

$$\int\frac{dx}{x(x^4-1)}=\int\frac{x^3\,dx}{x^4(x^4-1)}=\tfrac{1}{4}\int\frac{dt}{t(t-1)},\text{ where } t=x^4,$$

$$=\tfrac{1}{4}\int\left\{-\frac{1}{t}+\frac{1}{(t-1)}\right\}dt=\tfrac{1}{4}\log\frac{t-1}{t}=\tfrac{1}{4}\log\frac{x^4-1}{x^4}.$$

Ex. 2. Integrate $x^2/(x^2-1)(x^2+2)$.

$$\int\frac{x^2\,dx}{(x^2-1)(x^2+2)}=\int\left\{\frac{1}{3(x^2-1)}+\frac{2}{3(x^2+2)}\right\}dx=\text{etc.}$$

Ex. 3. Use the substitutions $x+1/x=t$, $x-1/x=u$ to solve the worked out Example 2 of § 2·6.

$$\int\frac{x^2\,dx}{x^4+x^2+1}=\int\frac{dx}{x^2+1+1/x^2}=\tfrac{1}{2}\int\frac{(1-1/x^2)\,dx}{x^2+1+1/x^2}+\tfrac{1}{2}\int\frac{(1+1/x^2)\,dx}{x^2+1+1/x^2}$$

$$=\tfrac{1}{2}\int\frac{(1-1/x^2)\,dx}{(x+1/x)^2-1}+\tfrac{1}{2}\int\frac{(1+1/x^2)\,dx}{(x-1/x)^2+3}.$$

Now put $x+1/x=t$ in the first integral and $x-1/x=u$ in the second; then $(1-1/x^2)\,dx=dt$ and $(1+1/x^2)\,dx=du$. So

$$\int\frac{x^2\,dx}{x^4+x^2+1}=\tfrac{1}{2}\int\frac{dt}{t^2-1}+\tfrac{1}{2}\int\frac{du}{u^2+3}$$

$$=\tfrac{1}{4}\log\frac{t-1}{t+1}+\frac{1}{2\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}}$$

$$=\tfrac{1}{4}\log\frac{x+1/x-1}{x+1/x+1}+\frac{1}{2\sqrt{3}}\tan^{-1}\frac{x-1/x}{\sqrt{3}}$$

$$=\tfrac{1}{4}\log\frac{x^2-x+1}{x^2+x+1}+\frac{1}{2\sqrt{3}}\tan^{-1}\frac{x^2-1}{\sqrt{3}x}.$$

[This answer differs from the answer obtained before merely by a constant, because

$$\tan^{-1}\{(x^2-1)/\sqrt{3}x\}=-\tan^{-1}\{(1-x^2)/\sqrt{3}x\}$$

$$=-\cot^{-1}\{\sqrt{3}x/(1-x^2)\}=-\tfrac{1}{2}\pi+\tfrac{1}{2}\pi-\cot^{-1}\{\sqrt{3}x/(1-x^2)\}$$

$$=-\tfrac{1}{2}\pi+\tan^{-1}\{\sqrt{3}x/(1-x^2)\}=-\tfrac{1}{2}\pi+\text{the previous result.}]$$

Ex. 4. Integrate $1/(x^{2n}+1)$.

The roots of $x^{2n}=-1$, i.e., $x^{2n}=\cos\pi+i\sin\pi$, are

$$\cos\frac{(2p+1)\pi}{2n}+i\sin\frac{(2p+1)\pi}{2n},$$

where p takes the values $0, 1, 2, \ldots, (2n-1)$. Hence, writing r for $2p+1$ and θ for $\pi/2n$, the roots are $\cos r\theta \pm i \sin r\theta$, where r takes the values $1, 3, 5, \ldots, (2n-1)$. Let α denote the root $\cos r\theta + i \sin r\theta$ and β the root $\cos r\theta - i \sin r\theta$. Then if we denote $x^{2n}+1$ by $\phi(x)$, the partial fraction corresponding to $x-\alpha$ is, by § 2·2,

$$\frac{1}{(x-\alpha)\phi'(\alpha)} = \frac{1}{2n\alpha^{2n-1}(x-\alpha)} = \frac{\alpha}{2n\alpha^{2n}(x-\alpha)} = \frac{\alpha}{-2n(x-\alpha)}.$$

Hence the sum of the partial fractions corresponding to α, β is

$$-\frac{1}{2n}\frac{(\alpha+\beta)x-2\alpha\beta}{x^2-(\alpha+\beta)x+\alpha\beta} = -\frac{1}{2n}\frac{2(x\cos r\theta-1)}{x^2-2x\cos r\theta+1}$$

$$= -\frac{1}{2n}\cos r\theta \frac{2x-2\cos r\theta}{x^2-2x\cos r\theta+1} + \frac{1}{2n}\frac{2-2\cos^2 r\theta}{(x-\cos r\theta)^2+\sin^2 r\theta}.$$

So $\int dx/(x^{2n}+1) = -(1/2n)\Sigma \cos r\theta \log(x^2-2x\cos r\theta+1)$
$+(1/n)\Sigma \sin r\theta \tan^{-1}\{(x-\cos r\theta)/\sin r\theta\}$,

where r takes up the values $1, 3, \ldots, 2n-1$, and $\theta=\pi/2n$.

EXAMPLES

Integrate

1. $1/(x^2+1)^3$.
2. $1/(2x^2+1)^2$.
3. $1/(x^2+x+1)^2$.
4. $(2x+3)/(x^2+2x+3)^2$.
5. $2x/(1+x)(1+x^2)^2$.
6. $x^4/(x^2+1)^2$.
7. $(x^2+4)/(x^2+1)(x^2+3)$.
8. $1/(x^2+a^2)(x^2+b^2)$.
9. $(x^2+1)/(x^4+1)$. [Put $x-1/x=t$.] [*Delhi*, 1959]
10. $(x^2-1)/(x^4+x^2+1)$. [Put $x+1/x=t$.] [*Allahabad*, 1962]
11. $(x^2+1)/(x^4-x^2+1)$.
12. $x^2/(x^4+a^4)$. [*Lucknow*,'57]
13. $1/(x^4+8x^2+9)$. [*Aligarh*, 1951]
14. $2x/(x^2+1)(x^2+3)$.
15. $1/x(x^2+1)^3$. [*Alld.*, '53]
16. $1/x(x^5+1)$. [*Raj.*, 1957]

Evaluate

17. $\int_1^\infty \frac{dx}{x(1+x^2)}$. [*Baroda*,'59]
18. $\int_2^3 \frac{x^2\,dx}{(x^2-1)(x^2+2)}$. [*Alld.*,'45]

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EXAMPLES ON CHAPTER II

Integrate

1. $1/(x^2-x-6)$.

2. $(x^2+x-1)/(x^3+x^2-6x)$. [*Panjab*, 1957]

Imp 3. $(x^2+1)/(x^2-1)^2$. [*Gorakhpur*, 1960]

4. $1/(x-1)^2(x-2)(x^2+4)$. [*Panjab*, 1954]

5. $1/(x-1)^2(x^2+1)^2$. [*Bom.*, '50]

6. $1/(2+x^2)(1-x^2)$.

7. $1/(x^4-1)$.

8. $1/(x^4+1)$. [*Raj.*, '60]

9. $(x+a)/x^2(x-a)(x^2+a^2)$.

10. $1/x(x^n+1)$. [*Kar.*, '59]

11. $x/(x^4+x^2+1)$. [*Delhi*, '58]

12. $(1-x^2)/x(1+x^2+x^4)$.

13. $x^5/(x^9-1)$.

14. $1/(x-1)^3(x^3+1)$. [*Gor.*, '60]

15. $(5x+3)/(2x^2+x+2)^2$.

16. $(x-a)/(x^2+a^2)^3$.

17. $(x^2-2)/(x^2+2)^3$.

18. $\displaystyle\int_1^2 \frac{dx}{x(1+2x)^2}$.

Evaluate

19. $\displaystyle\int_0^{\pi/4} \sqrt{(\tan\theta)}\, d\theta$. [Put $\tan\theta = t^2$.] [*Rajasthan*, 1961]

20. $\displaystyle\int_0^{\pi/4} \sqrt{(\cot\theta)}\, d\theta$. [*Agra*, 1955]

Imp 21. $\displaystyle\int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$. [*Gorakhpur*, 1960]

Imp 22. $\displaystyle\int_2^3 \frac{x^2+1}{(2x+1)(x-1)(x+1)}\, dx$. [*Allahabad*, 1958]

23. Show that

$$\int_0^\infty \frac{x^2\, dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2(a+b)(b+c)(c+a)}. \quad [\textit{Dacca}, \text{'45}]$$

24. Prove that

$$\int_{-\infty}^\infty \frac{dx}{(x^2+ax+a^2)(x^2+bx+b^2)} = \frac{2\pi(a+b)}{(\sqrt{3})\, ab(a^2+ab+b^2)}.$$

CHAPTER III

INTEGRATION OF IRRATIONAL ALGEBRAIC FRACTIONS

3·1. A rational function of a root of a linear expression and x. Rational functions of $(ax+b)^{1/n}$ and x can be easily evaluated by the substitution $t^n=ax+b$. Thus

$$\int f\{x, (ax+b)^{1/n}\}\,dx=\int f\left(\frac{t^n-b}{a},\ t\right)\frac{nt^{n-1}}{a}\,dt.$$

Since by supposition f is a rational function of x and $(ax+b)^{1/n}$, it follows that on the right hand the integrand is a rational function of t, and so the methods of the last chapter are applicable.

Rational functions of x, $(ax+b)^{1/n}$ and $(ax+b)^{1/m}$ can be similarly evaluated by the substitution $t^p=ax+b$, where p is the lowest common multiple of m and n.

Ex. 1. Integrate $x/(x-3)\sqrt{(x+1)}$. [*Baroda*, 1960]

Put $x+1=t^2$. We get

$$\int\frac{x\,dx}{(x-3)\sqrt{(x+1)}}=\int\frac{(t^2-1)2t\,dt}{(t^2-4)t}=2\int\frac{t^2-1}{t^2-4}\,dt$$

$$=2\int\left\{1+\frac{3}{(t^2-4)}\right\}dt$$

$$=2t+\tfrac{3}{2}\log\frac{t-2}{t+2}$$

$$=2\sqrt{(x+1)}+\tfrac{3}{2}\log\frac{\sqrt{(x+1)}-2}{\sqrt{(x+1)}+2}.$$

Ex. 2. Integrate $(1+x^{1/2})/(1+x^{1/3})$.

Putting $x=t^6$, we have

$$\int\frac{1+x^{1/2}}{1+x^{1/3}}\,dx=6\int\frac{1+t^3}{1+t^2}\,t^5\,dt=6\int\frac{t^8+t^5}{t^2+1}\,dt$$

$$=6\int\left\{t^6-t^4+t^3+t^2-t-1+\frac{t+1}{t^2+1}\right\}dt.$$

Therefore

$$\int\frac{1+x^{1/2}}{1+x^{1/3}}\,dx=\tfrac{6}{7}t^7-\tfrac{6}{5}t^5+\tfrac{3}{2}t^4+2t^3-3t^2-6t$$

$$+3\log(t^2+1)+6\tan^{-1}t, \text{ where } t=x^{1/6}.$$

EXAMPLES

Integrate

1. $x^2/\sqrt{(x+5)}$.
2. $\sqrt{x}/(1+x)$. [*Nagpur*, '53]
3. $x^3/\sqrt{(x-1)}$.
4. $(x+3)/(x-3)^{1/3}$.
5. $x^3/(x-1)\sqrt{(x+2)}$. [*Aligarh*, 1947]
6. $\sqrt{(x^2-a^2)}/x$.
7. $1/x\sqrt{(x^3+1)}$.
8. $(1+\sqrt{x})/(1+x^{1/4})$.
9. $x/\{(1+x)^{1/3}-(1+x)^{1/2}\}$.
10. Evaluate $\displaystyle\int_8^{15}\frac{dx}{(x-3)\sqrt{(x+1)}}$. [*Lucknow*, 1944]

3·2. Integration of $1/\sqrt{(ax^2+bx+c)}$. We can integrate $1/\sqrt{(ax^2+bx+c)}$ by throwing ax^2+bx+c into the form $a\{(x+\alpha)^2\pm\beta^2\}$ as in §2·5. The result will assume different forms according to the signs of a and β^2. In the following discussion it is supposed that the positive square root is taken everywhere.

Case I. a positive.

$$\int\frac{dx}{\sqrt{(ax^2+bx+c)}}=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\{(x+b/2a)^2+(c/a-b^2/4a^2)\}}}$$

$$\text{or }\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\{(x+b/2a)^2-(b^2/4a^2-c/a)\}}}.$$

If $c/a-b^2/4a^2$ is positive, i.e., $b^2<4ac$, we take the first form, and get

$$\int\frac{dx}{\sqrt{(ax^2+bx+c)}}=\frac{1}{\sqrt{a}}\sinh^{-1}\frac{x+b/2a}{\sqrt{(c/a-b^2/4a^2)}}$$

$$=\frac{1}{\sqrt{a}}\sinh^{-1}\frac{2ax+b}{\sqrt{(4ac-b^2)}}.$$

If $c/a-b^2/4a^2$ is negative, i.e., $b^2>4ac$, we take the second form and get

$$\int\frac{dx}{\sqrt{(ax^2+bx+c)}}=\frac{1}{\sqrt{a}}\cosh^{-1}\frac{2ax+b}{\sqrt{(b^2-4ac)}}.$$

Case II. a negative. Here $\sqrt{(-a)}$ is real. So

$$\int\frac{dx}{\sqrt{(ax^2+bx+c)}}=\frac{1}{\sqrt{(-a)}}\int\frac{dx}{\sqrt{\{-c/a-(x^2+bx/a)\}}}$$

$$=\frac{1}{\sqrt{(-a)}}\int\frac{dx}{\sqrt{\{(b^2/4a^2-c/a)-(x+b/2a)^2\}}}$$

$$=\frac{1}{\sqrt{(-a)}}\sin^{-1}\frac{x+b/2a}{\sqrt{(b^2/4a^2-c/a)}}$$

$$=\frac{1}{\sqrt{(-a)}}\sin^{-1}\frac{-2ax-b}{\sqrt{(b^2-4ac)}},$$

since the positive square root of $b^2/4a^2-c/a$ is $\sqrt{(b^2-4ac)}/(-a)$ when a is negative. Cf. Ex. 2 below.

Ex. 1. Integrate $1/\sqrt{(2x^2-x+2)}$.

$$\int\frac{dx}{\sqrt{(2x^2-x+2)}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{(x^2-\frac{1}{2}x+\frac{1}{16}+\frac{15}{16})}}$$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\{(x-\frac{1}{4})^2+(\sqrt{15}/4)^2\}}}$$

$$=\frac{1}{\sqrt{2}}\sinh^{-1}\frac{x-\frac{1}{4}}{\sqrt{15}/4}=\frac{1}{\sqrt{2}}\sinh^{-1}\frac{4x-1}{\sqrt{15}}.$$

Ex. 2. Integrate $1/\sqrt{(2+x-3x^2)}$.

$$\int\frac{dx}{\sqrt{(2+x-3x^2)}}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\{\frac{2}{3}-(x^2-\frac{1}{3}x)\}}}$$

$$=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\{\frac{2}{3}+\frac{1}{36}-(x^2-\frac{1}{3}x+\frac{1}{36})\}}}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\{\frac{25}{36}-(x-\frac{1}{6})^2\}}}$$

$$=\frac{1}{\sqrt{3}}\sin^{-1}\frac{x-\frac{1}{6}}{\frac{5}{6}}=\frac{1}{\sqrt{3}}\sin^{-1}\frac{6x-1}{5}.$$

3·21. Integration of $\sqrt{(ax^2+bx+c)}$. The integration of $\sqrt{(ax^2+bx+c)}$ can be accomplished by reducing ax^2+bx+c to the form $a\{(x+\alpha)^2\pm\beta^2\}$ as in the last article.

Ex. Integrate $\sqrt{(1+x-2x^2)}$.

$$\int\sqrt{(1+x-2x^2)}\,dx=\sqrt{2}\int\sqrt{\{\tfrac{1}{2}-(x^2-\tfrac{1}{2}x)\}}dx$$

$$=\sqrt{2}\int\sqrt{\{\tfrac{1}{2}+\tfrac{1}{16}-(x-\tfrac{1}{4})^2\}}\,dx=\sqrt{2}\int\sqrt{\{\tfrac{9}{16}-(x-\tfrac{1}{4})^2\}}\,dx$$

$$=\sqrt{2}\,.\,\tfrac{1}{2}(x-\tfrac{1}{4})\sqrt{\{\tfrac{9}{16}-(x-\tfrac{1}{4})^2\}}+\sqrt{2}\,.\,\tfrac{9}{32}\sin^{-1}\{(x-\tfrac{1}{4})/(\tfrac{3}{4})\}$$

$$=\tfrac{1}{2}(x-\tfrac{1}{4})\sqrt{(1+x-2x^2)}+\tfrac{9}{32}\{\sqrt{2}\sin^{-1}\tfrac{1}{3}(4x-1)\}.$$

3·3. Integration of $(px+q)/\sqrt{(ax^2+bx+c)}$. We can integrate $(px+q)/\sqrt{(ax^2+bx+c)}$ by breaking it up into two parts in one of which the numerator is the differential coefficient of ax^2+bx+c and in the other the numerator does not involve x. Thus

$$\int\frac{(px+q)\,dx}{\sqrt{(ax^2+bx+c)}}$$

$$=\frac{p}{2a}\int\frac{(2ax+b)\,dx}{\sqrt{(ax^2+bx+c)}}+\int\frac{(q-bp/2a)\,dx}{\sqrt{(ax^2+bx+c)}}.$$

The first integral on the right is evidently equal to $2(p/2a)\sqrt{(ax^2+bx+c)}$. The second can be evaluated by the method of § 3·2.

Evidently, $(px+q)\sqrt{(ax^2+bx+c)}$ also can be integrated by a similar method.

Ex. 1. Integrate $(x+1)/\sqrt{(x^2-x+1)}$.

$$\int \frac{(x+1)\,dx}{\sqrt{(x^2-x+1)}} = \tfrac{1}{2}\int \frac{(2x-1)\,dx}{\sqrt{(x^2-x+1)}} + \int \frac{(1+\frac{1}{2})\,dx}{\sqrt{(x^2-x+1)}}$$

$$= \sqrt{(x^2-x+1)} + \tfrac{3}{2}\int \frac{dx}{\sqrt{\{(x-\frac{1}{2})^2+\frac{3}{4}\}}}$$

$$= \sqrt{(x^2-x+1)} + \tfrac{3}{2}\sinh^{-1}\frac{x-\frac{1}{2}}{\sqrt{3}/2}$$

$$= \sqrt{(x^2-x+1)} + \tfrac{3}{2}\sinh^{-1}\frac{2x-1}{\sqrt{3}}.$$

Ex. 2. Integrate $(x+1)\sqrt{(x^2-x+1)}$.

$$\int (x+1)\sqrt{(x^2-x+1)}\,dx$$

$$= \tfrac{1}{2}\int (2x-1)\sqrt{(x^2-x+1)}\,dx + \int (1+\tfrac{1}{2})\sqrt{(x^2-x+1)}\,dx$$

$$= \tfrac{1}{2}\cdot\tfrac{2}{3}(x^2-x+1)^{3/2} + \tfrac{3}{2}\int \sqrt{\{(x-\tfrac{1}{2})^2+\tfrac{3}{4}\}}\,dx$$

$$= \tfrac{1}{3}(x^2-x+1)^{3/2} + \tfrac{3}{2}\cdot\tfrac{1}{2}(x-\tfrac{1}{2})\sqrt{\{(x-\tfrac{1}{2})^2+\tfrac{3}{4}\}}$$

$$+ \tfrac{3}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}\sinh^{-1}\frac{x-\frac{1}{2}}{\sqrt{3}/2}$$

$$= \tfrac{1}{24}(8x^2+10x-1)\sqrt{(x^2-x+1)} + \tfrac{9}{16}\sinh^{-1}\{(2x-1)/\sqrt{3}\}.$$

EXAMPLES

Integrate

1. $1/\sqrt{(x^2+2x+3)}$. 2. $1/\sqrt{(1-x-x^2)}$.
3. $1/\sqrt{(2x^2+3x+4)}$. 4. $\sqrt{(2x^2+3x+4)}$.
5. $\sqrt{(4-3x-2x^2)}$. 6. $x/\sqrt{(x^2+x+1)}$. [*Nag.*, '52]
7. $(2x+5)/\sqrt{(x^2+3x+1)}$. [*Banaras, Geophysics*, 1955]
8. $(x+1)\sqrt{(2x^2+3)}$. 9. $(3x-2)\sqrt{(x^2+x+1)}$.
10. Evaluate $\int_\alpha^\beta \sqrt{\{(x-\alpha)(\beta-x)\}}\,dx$. [*Utkal*, 1956]

Q-9.72

3·4. Integration of $(\mathbf{c_0x^n+c_1x^{n-1}+\ldots+c_n})/\sqrt{(\mathbf{ax^2+bx+c})}$.

To integrate $(c_0x^n+c_1x^{n-1}+\ldots+c_n)/\sqrt{(ax^2+bx+c)}$ we assume a suitable form for the result, differentiate both sides, and by comparing coefficients of various powers of x obtain the value of the unknown coefficients occurring in the assumed form. Thus, suppose

$$\int\frac{c_0x^n+c_1x^{n-1}+\ldots+c^n}{\sqrt{(ax^2+bx+c)}}\,dx$$
$$=(C_0x^{n-1}+C_1x^{n-2}+\ldots+C_{n-1})\sqrt{(ax^2+bx+c)}$$
$$+C_n\int\frac{dx}{\sqrt{(ax^2+bx+c)}},$$

where the C's are constants.

Differentiating both sides, and multiplying by $\sqrt{(ax^2+bx+c)}$, we have

$$c_0x^n+c_1x^{n-1}+\ldots+c_n=\{(n-1)C_0x^{n-2}$$
$$+(n-2)C_1x^{n-3}+\ldots+C_{n-2}\}(ax^2+bx+c)$$
$$+(C_0x^{n-1}+\ldots+C_{n-1})(ax+\tfrac{1}{2}b)+C_n.$$

Both sides are now rational integral functions of x. Equating the coefficients of like powers of x, we have $n+1$ equations which give us the values of the $n+1$ constants C_0, C_1, ..., C_n.

Also we know how to evaluate

$$\int\frac{dx}{\sqrt{(ax^2+bx+c)}}.$$

Hence the integral will be completely determined.

Ex. 1. Integrate $(6x^3+15x^2-7x+6)/\sqrt{(2x^2-2x+1)}$.

Let $$\int\frac{6x^3+15x^2-7x+6}{\sqrt{(2x^2-2x+1)}}dx$$
$$=(C_0x^2+C_1x+C_2)\sqrt{(2x^2-2x+1)}+C_3\int\frac{dx}{\sqrt{(2x^2-2x+1)}}.$$

Differentiating both sides with respect to x and multiplying by $\sqrt{(2x^2-2x+1)}$, we have
$$6x^3+15x^2-7x+6=(2C_0x+C_1)(2x^2-2x+1)\}$$
$$+(C_0x^2+C_1x+C_2).\tfrac{1}{2}(4x-2)+C_3.$$

Therefore $6C_0=6,\ -4C_0+2C_1+2C_1-C_0=15,$
$$2C_0-2C_1+2C_2-C_1=-7,\quad C_1-C_2+C_3=6.$$

Hence $C_0=1, C_1=5, C_2=3, C_3=4.$

Also $$\int\frac{dx}{\sqrt{(2x^2-2x+1)}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\{(x^2-x+\frac{1}{4})+\frac{1}{4}\}}}$$
$$=\frac{1}{\sqrt{2}}\sinh^{-1}\frac{x-\frac{1}{2}}{\frac{1}{2}}=\frac{1}{\sqrt{2}}\sinh^{-1}(2x-1).$$

So $$\int\frac{6x^3+15x^2-7x+6}{\sqrt{(2x^2-2x+1)}}dx=(x^2+5x+3)\sqrt{(2x^2-2x+1)}$$
$$+2\sqrt{2}\sinh^{-1}(2x-1).$$

Ex. 2. Integrate $(x^2+1)/\sqrt{(x^2+3)}$.

We can proceed as above, or more simply as follows:
$$\int\frac{(x^2+1)}{\sqrt{(x^2+3)}}dx=\int\frac{x^2+3-2}{\sqrt{(x^2+3)}}dx=\int\sqrt{(x^2+3)}\,dx-2\int\frac{dx}{\sqrt{(x^2+3)}}$$
$$=\tfrac{1}{2}x\sqrt{(x^2+3)}+\tfrac{3}{2}\sinh^{-1}(x/\sqrt{3})-2\sinh^{-1}(x/\sqrt{3})$$
$$=\tfrac{1}{2}x\sqrt{(x^2+3)}-\tfrac{1}{2}\sinh^{-1}(x/\sqrt{3}).$$

NOTE. This method can always be applied when the numerator is of the second degree.

EXAMPLES

Integrate

1. $(x^2-x+1)/\sqrt{(2x^2-x+2)}$.
2. $(x^2+1)/\sqrt{(x^2+4)}$. [*Nagpur*, 1952]

3. $(x^2-2)/\sqrt{(3-x^2)}$. [*Sagar*, 1954]

4. $(x^2+2x+3)/\sqrt{(x^2+x+1)}$. [*Rajputana*, 1950]

5. $(x^3+3)/\sqrt{(x^2+1)}$. [*P.S.C., U.P., Forest*, 1955]

Imp

6. Evaluate $\displaystyle\int_0^1 \frac{1-4x+2x^2}{\sqrt{(2x-x^2)}}\,dx.$ [*Allahabad*, 1950]

Imp

3·5. Integration of $1/(x-k)^r\sqrt{(ax^2+bx+c)}$.

The substitution $x-k=1/t$ reduces the integration of $1/(x-k)^r\sqrt{(ax^2+bx+c)}$ to the problem of integrating an expression of the form $t^{r-1}/\sqrt{(At^2+Bt+C)}$. It is supposed that $x>k$, so that t is positive. If $x<k$, it is best to put $k-x=1/t$, so that t is positive again.

Supposing now that $x>k$, and $x-k=1/t$, we have

$$\int\frac{dx}{(x-k)^r\sqrt{(ax^2+bx+c)}}=\int\frac{-(1/t^2)\,dt}{(1/t)^r\sqrt{(ax^2+bx+c)}}$$

$$=-\int\frac{t^{r-1}\,dt^*}{\sqrt{(ax^2t^2+bxt^2+ct^2)}}$$

$$=-\int\frac{t^{r-1}\,dt}{\sqrt{\{a(1+kt)^2+bt(1+kt)+ct^2\}}}$$

$$=-\int\frac{t^{r-1}\,dt}{\sqrt{\{(ak^2+bk+c)t^2+(2ak+b)t+a\}}}$$

$$=-\int\frac{t^{r-1}\,dt}{\sqrt{\{At^2+Bt+C\}}},$$

where $A=ak^2+bk+c$, $B=2ak+b$, $C=a$.

*If t is negative, this expression will not be equal to the previous one; we must, in fact, change its sign. The reason is that we have multiplied the denominator by $\sqrt{t^2}$, and so the numerator must be multiplied by t or $-t$, whichever is positive. The student will have no difficulty about signs in numerical examples.

This can be integrated by the method of § 3·2 if $r=1$, or by the method of § 3·4 if $r>1$.

Ex. Integrate $1/(x-1)\sqrt{(x^2+x+1)}$, $x>1$.

$$\int\frac{dx}{(x-1)\sqrt{(x^2+x+1)}}=-\int\frac{(1/t^2)\,dt}{(1/t)\sqrt{(x^2+x+1)}},$$

putting $x-1=1/t$ and $dx=-(1/t^2)\,dt$,

$$=-\int\frac{dt}{\sqrt{(t^2x^2+t^2x+t^2)}}=-\int\frac{dt}{\sqrt{\{(t+1)^2+t(t+1)+t^2\}}}$$

$$=-\int\frac{dt}{\sqrt{(3t^2+3t+1)}}=-\frac{1}{\sqrt{3}}\int\frac{dt}{\sqrt{\{(t+\frac{1}{2})^2+\frac{1}{3}-\frac{1}{4}\}}}$$

$$=-\frac{1}{\sqrt{3}}\sinh^{-1}\frac{t+\frac{1}{2}}{\sqrt{(1/12)}}=-\frac{1}{\sqrt{3}}\sinh^{-1}\left\{(\sqrt{3})\left(\frac{x+1}{x-1}\right)\right\}.$$

NOTE. If $f(x)$ is a rational fraction and its denominator can be resolved into real linear factors, it is obvious that $f(x)/\sqrt{(ax^2+bx+c)}$ can be integrated by first resolving $f(x)$ into partial fractions.

EXAMPLES

Integrate the following, supposing the integrand to be positive :

1. $1/(x-1)\sqrt{(x^2+1)}$. [*Andhra*, 1941]
2. $1/(1+x)\sqrt{(1+x-x^2)}$. [*Andhra*, 1960]
3. $1/(x+1)\sqrt{(1+2x-x^2)}$. [*I. A. S.*, 1960]
4. $1/(x-a)\sqrt{(x^2-a^2)}$. [*Nagpur*, 1956]
5. $1/(x^2-1)\sqrt{(1+x^2)}$. [*Allahabad*, 1940]

   [Break up $1/(x^2-1)$ into its partial fractions.]
6. $1/(x-1)^2\sqrt{(1-x^2)}$.
7. $1/x(x+1)\sqrt{(x^2+x-1)}$.
8. Show that $\displaystyle\int_1^2\frac{dx}{(x+1)\sqrt{(x^2-1)}}=\frac{1}{\sqrt{3}}$. [*Rajasthan*, 1962]

3·6. General case. Expressions which are not already in one of the forms previously considered can often be easily changed into an integrable form by rationalising the numerator or the denominator. Some expressions can be broken into two or more, each of which is integrable or can easily be reduced to an integrable form. The following discussion shows how this can be effected in the case of a rational function of x and $\sqrt{(ax^2+bx+c)}$. Article 3·71 will show that any such function can always be integrated in terms of the elementary functions alone.

Let $f(x, \sqrt{X})$ be a rational function of x and $\sqrt{X}$, where $X \equiv ax^2+bx+c$. Then, since every even power of $\sqrt{X}$ is a rational integral function of x, and every odd power of $\sqrt{X}$ is equal to $\sqrt{X}$ multiplied by a rational integral function of x, the most general form for $f(x, \sqrt{X})$ is

$$\frac{P+Q\sqrt{X}}{R+S\sqrt{X}}, \quad . \quad . \quad . \quad . \quad (1)$$

where P, Q, R and S are rational integral functions of x. Rationalising the denominator by multiplying by $R-S\sqrt{X}$, we find that (1) reduces to

$$\frac{PR-QSX+(RQ-PS)\sqrt{X}}{R^2-S^2X},$$

i.e., to

$$\frac{PR-QSX}{R^2-S^2X}+\frac{(RQ-PS)X}{(R^2-S^2X)\sqrt{X}}.$$

We can integrate the first function (which does not involve $\sqrt{X}$) by methods applicable to rational functions. To integrate the second part, we can break up $(RQ-PS)X \div (R^2-S^2X)$ into partial fractions. Then we shall have to integrate terms of the type

$$\frac{a_0x^n+a_1x^{n-1}+\ldots+a_n}{\sqrt{X}}, \quad \frac{1}{(x-\alpha)\sqrt{X}}, \quad \frac{1}{(x-\alpha)^r\sqrt{X}},$$

$$\frac{AX+B}{(x^2+\alpha x+\beta)\sqrt{X}}, \quad \frac{Ax+B}{(x^2+\alpha x+\beta)^r\sqrt{X}}.$$

The integration of the first three of these forms has been considered before. The last two forms occur only rarely, and can be dealt with by the method of § 3·71 when they occur, but an easy special case can be dealt with in a different way, as shown in § 3·7.

Ex. Integrate $\sqrt{(x+1)}/(x+2)\sqrt{(x+3)}$.

$$\int\frac{\sqrt{(x+1)}\,dx}{(x+2)\sqrt{(x+3)}}=\int\left\{\frac{x+1}{x+2}\cdot\frac{dx}{\sqrt{\{(x+1)(x+3)\}}}\right\}$$
$$=\int\left(1-\frac{1}{x+2}\right)\frac{dx}{\sqrt{(x^2+4x+3)}}$$
$$=\int\frac{dx}{\sqrt{(x^2+4x+3)}}-\int\frac{dx}{(x+2)\sqrt{(x^2+4x+3)}}.$$

These integrals can now be evaluated by the methods of §§ 3·2 and 3·5.

3·7. Integration of $1/(\mathbf{A}x^2+\mathbf{B})\sqrt{(\mathbf{C}x^2+\mathbf{D})}$.

To integrate $1/(Ax^2+B)\sqrt{(Cx^2+D)}$ we first put $x=1/t$. Thus

$$\int\frac{dx}{(Ax^2+B)\sqrt{(Cx^2+D)}}=\int\frac{-(1/t^2)\,dt}{(A/t^2+B)\sqrt{(C/t^2+D)}}$$
$$=-\int\frac{t\,dt}{(A+Bt^2)\sqrt{(C+Dt^2)}}.$$

Now the substitution $C+Dt^2=u^2$ reduces it to the form $\int du/(u^2\pm a^2)$.

Ex. Integrate $1/(1+x^2)\sqrt{(1-x^2)}$. [*Vikram*, 1961]

Putting $x=1/t$ and $dx=-(1/t^2)\,dt$, we have

$$\int\frac{dx}{(1+x^2)\sqrt{(1-x^2)}}=\int\frac{-(1/t^2)\,dt}{(1+1/t^2)\sqrt{(1-1/t^2)}}=-\int\frac{t\,dt}{(t^2+1)\sqrt{(t^2-1)}}$$
$$=-\frac{u\,du}{(u^2+2)u},\text{ putting } t^2-1=u^2 \text{ and } t\,dt=u\,du$$
$$=-\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}=-\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{(t^2-1)}}{\sqrt{2}}$$
$$=-\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{(1-x^2)}}{x\sqrt{2}}.$$

EXAMPLES

Integrate

1. $\sqrt{\{(1+x)/(1-x)\}}$. [*Utkal*, 1951]
2. $\sqrt{\{(x+1)/(x-1)\}}$.
3. $\{1+x+\sqrt{(1+x^2)}\}/\{1+x-\sqrt{(1+x^2)}\}$.
4. $x\sqrt{\{(1-x)/(1+x)\}}$. 5. $1/\{x+\sqrt{(x^2-1)}\}$.
6. $1/(2x^2+3)\sqrt{(x^2-4)}$. [*Utkal*, 1950]
7. $(x+1)/(x^2+4)\sqrt{(x^2+9)}$. [*I. A. S.*, 1953]
8. $1/(x^2+1)\sqrt{(x^2-1)}$. [*Delhi, Honours*, 1960]

3·71. Integral of a rational function of x and $\sqrt{(\mathbf{ax^2+bx+c})}$. By taking out $\sqrt{a}$ or $\sqrt{(-a)}$, whichever is real, as a factor, we can write the other factor of $\sqrt{(ax^2+bx+c)}$ in one of the forms

$$\sqrt{(x^2+hx+k)},$$

or

$$\sqrt{(-x^2+px+q)}.$$

Hence we need consider the integration of rational functions of x and one of the above two expressions only. We shall show that in each case, by a suitable substitution, the problem can be reduced to that of integrating a rational function of the new variable.

I. The substitution

$$x+\sqrt{(x^2+hx+k)}=t$$

will transform a rational function of x and $\sqrt{(x^2+hx+k)}$ into a rational function of t.

For, transposing x to the right, squaring, and solving for x, we get

$$x=\frac{t^2-k}{h+2t},\quad \frac{dx}{dt}=\frac{2(t^2+ht+k)}{(h+2t)^2},$$

and

$$\sqrt{(x^2+hx+k)}=t-\frac{t^2-k}{h+2t}.$$

It is obvious, therefore, that the new integral will involve only a rational function of t.

II. If $x^2-px-q\equiv(x-\alpha)(x-\beta)$, the substitution

$$\sqrt{(-x^2+px+q)}=(x-\alpha)t$$

will transform a rational function of x and $\sqrt{(-x^2+px+q)}$ into a rational function of t.

For on squaring, replacing $-x^2+px+q$ by $-(x-\alpha)(x-\beta)$, cancelling out $(x-\alpha)$, and solving for x, we get

$$x=\frac{\alpha t^2+\beta}{t^2+1},\quad \frac{dx}{dt}=\frac{2(\alpha-\beta)t}{(t^2+1)^2},$$

and
$$\sqrt{(-x^2+px+q)}=\frac{(\beta-\alpha)t}{t^2+1}.$$

It is obvious that the new integral in this case also will involve only a rational function of t.

NOTE 1. The above investigation proves that any rational function of x and $\sqrt{(ax^2+bx+c)}$ can be integrated in terms of the elementary functions, and only one or the other of the two substitutions given above need be used. All the forms involving $\sqrt{(ax^2+bx+c)}$ which have been considered before are rational functions of x and $\sqrt{(ax^2+bx+c)}$, and therefore can be integrated by the method of the present article also. But in actual practice the methods given earlier, if they are applicable, are far more expeditious.

2. The roots of $-x^2+px+q=0$, viz., α and β, must be real; for otherwise $-x^2+px+q$ will, for all values of x, have the same sign as the coefficient of x^2, i.e., will be negative, and so $\sqrt{(-x^2+px+q)}$ will be imaginary for all values of x.

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3·8. Integration of $\mathbf{x}^m(\mathbf{a}+\mathbf{b}\mathbf{x}^n)^p$.

The evaluation of

$$\int x^m(a+bx^n)^p\,dx,$$

where m, n and p are not necessarily integers, can be easily effected in three cases. The method of successive reduction also is applicable in these cases (as well as in some other cases), as shown in the next article.

I. p a positive integer.

In this case we can expand $(a+bx^n)^p$ by the Binomial Theorem into a finite series. Thus the integrand is resolved into the sum of a finite number of terms, each of which is easily integrable.

II. $(m+1)/n$ an integer.

Let $(m+1)/n=j+1$, where j is zero or an integer. Then $m=jn+(n-1)$, and the integral under consideration can be written as

$$\int x^{n-1}.\ (x^n)^j\ (a+bx^n)^p\, dx.$$

It is evident that by putting x^n equal to t we can reduce this to the case where only a linear function of the variable is raised to a fractional power (§ 3·1). Therefore, if $p=r/s$, the proper substitution after this will be to put $a+bt=u^s$. We can combine the two substitutions into one by putting directly

$$a+bx^n=u^s,$$

which gives $bnx^{n-1}\,dx=su^{s-1}\,du$. The integral will then be equal to

$$\frac{s}{bn}\int\left(\frac{u^s-a}{b}\right)^j u^{ps+s-1}\,du,$$

which can be easily evaluated by expanding $(u^s-a)^j$ by the Binomial Theorem if j is positive, or by the method of partial fractions if j is negative.

III. $p+(m+1)/n$ an integer, p not an integer.

In this case put $x=1/t$. Then the integral becomes

$$-\int\frac{1}{t^{m+2}}\left(a+\frac{b}{t^n}\right)^p dt,\ \text{i.e.,}\ -\int t^{-(m+pn+2)}(b+at^n)^p\,dt.$$

Hence this will come under Case II if $-(m+np+1)/n$ is an integer, i.e., if $p+(m+1)/n$ is an integer.

Ex. Integrate $x^{-2/3}(1+x^{1/2})^{-5/3}$.

Comparing with $x^m(a+bx^n)^p$, we find that here $p+(m+1)/n$ is an integer. Putting $x=1/t$, we have

$$\int x^{-2/3}(1+x^{1/2})^{-5/3}\,dx=-\int t^{(2/3)+(5/6)-2}\,(t^{1/2}+1)^{-5/3}\,dt$$

$$=-\int t^{-1/2}(1+t^{1/2})^{-5/3}\,dt.$$

Putting $1+t^{1/2}=u^3$, and $\frac{1}{2}t^{-1/2}\,dt=3u^2\,du$, we find that the integral

$$=-6\int u^{-5}u^2\,du=3u^{-2}=3(1+t^{1/2})^{-2/3}$$
$$=3(1+x^{-1/2})^{-2/3}.$$

3·81. Reduction formulae for $\int \mathbf{x}^m(\mathbf{a}+\mathbf{b}\mathbf{x}^n)^p\,\mathbf{dx}$. The integral $\int x^m(a+bx^n)^p\,dx$ can be connected with any one of the following six integrals :

(i) $\displaystyle\int x^{m-n}(a+bx^n)^p\,dx$, (ii) $\displaystyle\int x^m(a+bx^n)^{p-1}\,dx$

(iii) $\displaystyle\int x^{m+n}(a+bx^n)^p\,dx$, (iv) $\displaystyle\int x^m(a+bx^n)^{p+1}\,dx$,

(v) $\displaystyle\int x^{m-n}(a+bx^n)^{p+1}\,dx$, (vi) $\displaystyle\int x^{m+n}(a+bx^n)^{p-1}\,dx$.

We get thus six reduction formulae, the first two of which can be obtained, as shown below, by integrating by parts, breaking the new integral into two, transposing one and dividing by a constant. The third can be obtained from the first by writing $m+n$ for m and the fourth from the second by writing $p+1$ for p. The last two formulae can be

obtained at once by integrating by parts. (See equations (1) and (2) below).

(i) Integrating by parts, we have

$$\int x^m(a+bx^n)^p\,dx=\frac{1}{nb}\int x^{m-n+1}\,.\,nbx^{n-1}(a+bx^n)^p\,dx$$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}-\frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)(a+bx^n)^p\,dx \quad \ldots \quad (1)$$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}-\frac{m-n+1}{nb(p+1)}\,.\,a\int x^{m-n}(a+bx^n)^p\,dx$$

$$-\frac{m-n+1}{nb(p+1)}\,.\,b\int x^m(a+bx^n)^p\,dx.$$

Transposing the last term to the left and dividing by

$$1+(m-n+1)/n(p+1),$$

we get the required reduction formula :

$$\int x^m(a+bx^n)^p\,dx$$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(np+m+1)}-\frac{a(m-n+1)}{b(np+m+1)}\int x^{m-n}(a+bx^n)^p\,dx.$$

(ii) Again,

$$\int x^m(a+bx^n)^p\,dx=\frac{x^{m+1}(a+bx^n)^p}{m+1}-\frac{pbn}{m+1}\int x^{m+n}(a+bx^n)^{p-1}\,dx \quad \ldots \quad (2)$$

$$=\frac{x^{m+1}(a+bx^n)^p}{m+1}-\frac{pn}{m+1}\int x^m\{-a+(a+bx^n)\}(a+bx^n)^{p-1}\,dx.$$

It is easy now to break up the integral on the right into two, transpose one to the left and obtain another reduction formula for $\int x^m(a+bx^n)^p\,dx$ by division by a constant. We get

$$\int x^m(a+bx^n)^p\,dx=\frac{x^{m+1}(a+bx^n)^p}{np+m+1}$$

$$+\frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}\,dx.$$

(iii) The remaining reduction formulae can be easily obtained by the methods indicated above.

Examples

Integrate

1. $x^{2/3}(1+x^{6/5})^3$.
2. $x^3(1+x^2)^{1/3}$. [*Rajas.*, '62]
3. $x^{2n-1}(a+bx^n)^p$.
4. $x(1+x^3)^{1/3}$.
5. $x^2/(1+2x^4)^{3/4}$.
6. $x^{2n+1}/(a+bx^2)^{r/2}$, n and r being integers.
7. Prove that

By parts

$$\int (a^2+x^2)^{n/2}\,dx = \frac{x(a^2+x^2)^{n/2}}{n+1} + \frac{na^2}{n+1}\int (a^2+x^2)^{n/2-1}\,dx.$$

[*Aligarh*, 1955]

8. Apply the method of *reduction formulae* to find

$$\int (x^2+a^2)^{5/2}\,dx.$$ [*Aligarh*, 1960]

[Find first a reduction formula for $\int (a^2+x^2)^{n/2}\,dx$ by taking unity as the second function and integrating by parts].

9. If $I_n = \int x^n(a-x)^{1/2}\,dx$, prove that

$$(2n+3)I_n = 2anI_{n-1} - 2x^n(a-x)^{3/2}.$$

Evaluate $\int_0^a x^2\sqrt{(ax-x^2)}\,dx.$ [*Delhi*, 1960]

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10. If m be a positive integer, find a reduction formula for

$$\int x^m\sqrt{(2ax-x^2)}\,dx.$$ [*P. S. C., U. P.*, 1959]

Hence obtain the value of

$$\int_0^{2a} x^3\sqrt{(2ax-x^2)}\,dx.$$ [*Rajasthan*, '60]

[Hint. Notice that $x^m\sqrt{(2ax-x^2)} = x^{m+1/2}\sqrt{(2a-x)}$.]

11. If $U_n = \int x^n\sqrt{(a^2-x^2)}\,dx$, prove that

$$U_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{n+2} + \frac{n-1}{n+2}a^2U_{n-2}.$$

Evaluate $\int_0^a x^4\sqrt{(a^2-x^2)}\,dx.$ [*Poona*, 1957]

12. Investigate a formula of reduction applicable to

$$\int x^m(1+x^2)^{n/2}\,dx,$$

when m and n are positive integers, and complete the integration if $m=5$, $n=7$.

3·9. Substitutions. Functions involving $\sqrt{(a^2-x^2)}$, $\sqrt{(a^2+x^2)}$, or $\sqrt{(x^2-a^2)}$, and no other radical, can often be most conveniently integrated by a trigonometrical substitution. We can put $x=a\sin\theta$, or $a\tan\theta$, or $a\sec\theta$ respectively in the above cases and we shall get rid of the square root. Many examples to which this method is applicable will be given in the next chapter. Some simple ones are given below.

Since $\sqrt{(ax^2+bx+c)}$ can be easily reduced to one of the above forms, trigonometrical substitutions are applicable also in the case of functions involving $\sqrt{(ax^2+bx+c)}$. In cases where some power of x, say x^{n-1} is a factor of the integrand, and the remaining part is a function of x^n alone, the substitution of a new variable for x^n will often simplify the integration a great deal. The student should be on the lookout for such cases.

Ex. 1. Integrate $1/x^2\sqrt{(1+x^2)}$.

Putting $x=\tan\theta$, we get

$$\int\frac{dx}{x^2\sqrt{(1+x^2)}}=\int\frac{\sec^2\theta\,d\theta}{\tan^2\theta\,.\sec\theta}=\int\frac{\cos\theta\,d\theta}{\sin^2\theta}=-\operatorname{cosec}\theta$$
$$=-\sqrt{(1+x^2)}/x.$$

Ex. 2. Integrate $x^5/\sqrt{(1+x^3+x^6)}$.

Putting $x^3=t$, we get

$$\int\frac{x^5\,dx}{\sqrt{(1+x^3+x^6)}}=\tfrac{1}{3}\int\frac{t\,dt}{\sqrt{(1+t+t^2)}}$$
$$=\tfrac{1}{3}\sqrt{(1+x^3+x^6)}-\tfrac{1}{6}\sinh^{-1}\{(2x^3+1)/\sqrt{3}\}.$$

EXAMPLES

Integrate

1. $1/(a^2-b^2x^2)^{3/2}$.
2. $1/(a^2+b^2x^2)^{3/2}$.
3. $1/(a^2x^2-b^2)^{3/2}$.
4. $1/x\sqrt{(a^n+x^n)}$.
5. $(x^2+1)/x\sqrt{(1+x^4)}$.
6. $1/x^3\sqrt{(x^2-1)}$.

substitution

7. Evaluate $\displaystyle\int_0^a \frac{x\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}}\,dx$. [*Banaras, Eng.*, '58]

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EXAMPLES ON CHAPTER III

1. Integrate $1/(x+b)\sqrt{(x+a)}$. [*Gorakhpur*, 1960]

2. Evaluate $\displaystyle\int_0^c \frac{y\,dy}{\sqrt{(y+c)}}$.

Integrate

3. $(2-3x)/x\sqrt{(1+x)}$. [*Nagpur*, 1953]
4. $(2x^2+3)/\sqrt{(3-2x-x^2)}$.
5. $(x+a)/\sqrt{(x^2+b^2)}$.
6. $1/(x-a)\sqrt{\{(x-a)(b-x)\}}$.
7. $1/(1+x)\sqrt{(1-x^2)}$. [*Lucknow*, 1962]
8. $1/x^2\sqrt{(1+x^2)}$. [*Dacca*, 1940]
9. $\sqrt{\{(a-x)/x\}}$.
10. $1/\{\sqrt{(1+x)}+\sqrt{x}\}$. [*Allahabad*, 1962]

11. Evaluate $\displaystyle\int_2^3 \frac{dx}{(5x-6-x^2)^{1/2}}$.

12. Evaluate $\displaystyle\int \frac{dx}{\sqrt{\{(x-\alpha)(x-\beta)\}}}$. [*Ujjain*, 1960]

[Hint. One method is to proceed as in § 3·2. An easier method, however, is to put $x-\alpha=t^2$. Then the integral reduces to $\int 2\,dt/\sqrt{(t^2+\alpha-\beta)}$.]

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13. Evaluate $\displaystyle\int \sqrt{\left(\frac{x+a}{x+b}\right)}\cdot\frac{dx}{(x+c)}$.

[Rationalise the numerator and apply § 3·5, note.]

14. Prove that

$$\int_a^{\infty} \frac{dx}{x^4(a^2+x^2)^{1/2}} = \frac{2-\sqrt{2}}{3a^4}. \qquad \textit{[Banaras, 1958]}$$

15. Integrate $\sqrt{(1+x+x^2)}/(x+1)$.

16. Show that

x = 1/t

$$\int_0^{1/\sqrt{3}} \frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = \frac{\pi}{4\sqrt{2}}. \qquad \textit{[Allahabad, '62]}$$

17. Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{(1+x^2)^{n+1/2}}$.

18. Connect $\int x^{m-1}(a+bx^n)^p\,dx$ with $\int x^{m-n-1}(a+bx^n)^p\,dx$ and evaluate $\displaystyle\int \frac{x^8\,dx}{(1-x^3)^{1/3}}$. [*Second part, Banaras*, '57]

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19. If I_n denotes $\int_0^1 x^p(1-x^q)^n\,dx$, where p, q and n are positive, prove that

$$(qn+p+1)I_n = qn\,I_{n-1}.$$

Evaluate I_n when n is a positive integer. [*Panjab*, '44]

20. Prove that

$$\int_0^1 x^{-1/4}(1-x^{1/2})^{5/2}\,dx = \tfrac{5}{16}\int_0^1 x^{-1/4}(1-x^{1/2})^{1/2}\,dx.$$

[*Allahabad*, 1950]

CHAPTER IV

INTEGRATION OF TRANSCENDENTAL FUNCTIONS

4·1. Integration of $\sin^m x \cos^n x$. In every case in which m and n are positive integers,

$$\int \sin^m x \cos^n x\,dx$$

may be evaluated by the method of successive reduction, or by expressing $\sin^m x \cos^n x$ as the sum of sines or cosines of multiples of x. But if m or n is an odd positive integer, or if $m+n$ is an even negative integer, the integral can be evaluated more easily by a substitution, as shown below.

I. m or n an odd positive integer.

Let m be equal to $2r+1$, where r is zero or a positive integer; then we can integrate $\sin^m x \cos^n x$ by putting $\cos x = t$, whatever n may be. Thus

$$\int \sin^m x \cos^n x \, dx = \int \sin^{2r} x \cos^n x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^r \cos^n x \sin x \, dx$$

$$= -\int (1 - t^2)^r t^n \, dt, \text{ where } t = \cos x,$$

which is easy to evaluate by expanding $(1-t^2)^r$ by the Binomial Theorem.

Similarly, if n is an odd positive integer, we can put $\sin x = t$.

Ex. 1. Integrate $\sin^7 x$. [*Baroda*, 1960]

$$\int \sin^7 x \, dx = \int (1 - \cos^2 x)^3 \sin x \, dx = -\int (1 - t^2)^3 \, dt, \text{ where } t = \cos x,$$

$$= -\int (1 - 3t^2 + 3t^4 - t^6) \, dt = -t + t^3 - \tfrac{3}{5} t^5 + \tfrac{1}{7} t^7$$

$$= -\cos x + \cos^3 x - \tfrac{3}{5} \cos^5 x + \tfrac{1}{7} \cos^7 x.$$

Ex. 2. Integrate $\sin^{5/6} x \cos^3 x$.

$$\int \sin^{5/6} x \cos^3 x \, dx = \int \sin^{5/6} x (1 - \sin^2 x) \cos x \, dx$$

$$=\int t^{5/6}(1-t^2)\,dt, \text{ where } t=\sin x,$$

$$=\int (t^{5/6}-t^{17/6})\,dt=\tfrac{6}{11}t^{11/6}-\tfrac{6}{23}t^{23/6}$$

$$=\tfrac{6}{11}\sin^{11/6}x-\tfrac{6}{23}\sin^{23/6}x.$$

II. $m+n$ an even negative integer. (It is not necessary that m and n be integers).

Let $m+n=-2r$, where r is a positive integer; then we can integrate $\sin^m x\cos^n x$ by putting $\tan x=t$. Thus

$$\int \sin^m x\cos^n x\,dx=\int \tan^m x\cos^{m+n}x\,dx$$

$$=\int \tan^m x\sec^{2r}x\,dx=\int \tan^m x\sec^{2(r-1)}x\sec^2 x\,dx$$

$$=\int t^m(1+t^2)^{r-1}\,dt, \text{ where } t=\tan x,$$

which is easy to evaluate by expanding $(1+t^2)^{r-1}$ by the Binomial Theorem.

Ex. 1. Integrate $1/\sin^3 x\cos^5 x$. [*Allahabad*, 1953]

$$\int \frac{dx}{\sin^3 x\cos^5 x}=\int \frac{\sec^8 x}{\tan^3 x}\,dx=\int \frac{(1+\tan^2 x)^3\sec^2 x}{\tan^3 x}\,dx$$

$$=\int \frac{(1+t^2)^3\,dt}{t^3}, \text{ where } t=\tan x,$$

$$=\int \left(\frac{1}{t^3}+\frac{3}{t}+3t+t^3\right)dt$$

$$=-\tfrac{1}{2}t^{-2}+3\log t+\tfrac{3}{2}t^2+\tfrac{1}{4}t^4$$

$$=-\tfrac{1}{2}\cot^2 x+3\log\tan x+\tfrac{3}{2}\tan^2 x+\tfrac{1}{4}\tan^4 x$$

Ex. 2. Integrate $\sec^{2/3}x\,\mathrm{cosec}^{4/3}x$. [*Baroda*, 1960]

$$\int \sec^{2/3}x\,\mathrm{cosec}^{4/3}x\,dx=\int \frac{\sec^{2/3}x\,dx}{\sin^{4/3}x}$$

$$=\int \frac{\sec^2 x\,dx}{\tan^{4/3}x}=-3\tan^{-1/3}x.$$

4·11. Reduction formulae for $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$. The reduction formula for $\int \sin^n x\, dx$ has been given before (§ 1·37). The reduction formula for $\int \cos^n x\, dx$ may be obtained similarly, or from the one for $\int \sin^n x\, dx$ by writing $x+\frac{1}{2}\pi$ for x. The formulae are

$$\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx;$$

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

4·12. Reduction formula for $\int \sin^m x \cos^n x\, dx$.

$$\int \sin^m x \cos^n x\, dx = \int \sin^m x \cos x \cos^{n-1} x\, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+1} x \cos^{n-2} x \sin x\, dx,$$

on integration by parts,

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x(1-\cos^2 x)\, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\, dx$$

$$- \frac{n-1}{m+1}\int \sin^m x \cos^n x\, dx.$$

Transposing the last term to the left and dividing by $1+(n-1)/(m+1)$, i. e., by $(m+n) \div (m+1)$, we get the reduction formula

$$\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n}$$

$$+ \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\, dx.$$

If n is even, by repeatedly using this formula we shall reduce $\int \sin^m x \cos^n x\, dx$ to $\int \sin^m x\, dx$, which can be evaluated by § 4·11.

NOTE. By writing

$$\int \sin^m x \cos^n x\, dx \quad \text{as} \quad \int \sin^{m-1} x \cos^n x \sin x\, dx$$

and integrating by parts we can obtain the reduction formula

$$\int \sin^m x \cos^n x\, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx,$$

which diminishes the power of $\sin x$ instead of that of $\cos x$.

By writing $n+2$ for n in the first formula, and dividing by $(n+1)/(m+n+2)$ we get the reduction formula

$$\int \sin^m x \cos^n x\, dx$$

$$= -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x \cos^{n+2} x\, dx.$$

Similarly we can connect

$$\int \sin^m x \cos^n x\, dx \quad \text{with} \quad \int \sin^{m+2} x \cos^n x\, dx$$

by writing $m+2$ for m in the second formula. The last two formulae may prove useful when m or n is negative.

Notice that in the above, on integration by parts, we get an equation connecting $\int \sin^m x \cos^n x\, dx$ with either $\int \sin^{m+2} x \cos^{n-2} x\, dx$ or $\int \sin^m x \cos^{n+2} x\, dx$. These are also reduction formulae.

4·13. The integral $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$.

It is convenient to quote the value of this integral in terms of the Gamma Function. We shall not require the value of $\Gamma(x)$—read as Gamma x—for values of x other than a positive integer or half an odd positive integer. So the Gamma Function is sufficiently defined for us by the three equations

$$\Gamma(p+1) = p\,\Gamma(p), \qquad . \quad . \quad (1)$$

and $$\Gamma(1) = 1, \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \qquad . \quad . \quad (2)$$

Repeated applications of (1) will enable us to express the value of any Gamma Function in terms of $\Gamma(\theta)$, where θ lies between 0 and 1; and because $\Gamma(1)$ and $\Gamma(\frac{1}{2})$ are given numerically by (2), we shall be able to determine the numerical value of every $\Gamma(x)$ we shall come across.

In terms of the Gamma Function the value of the integral is given by

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}.$$

We shall verify the truth of this important result by applying the formula of the last article, viz.,

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x \, dx, \text{ which gives}$$

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \left[\frac{\sin^{m+1} x \cos^{n-1} x}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x \, dx$$

$$= \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x \, dx, \quad (3)$$

and the formula of § 4·11, which gives

$$\int_0^{\pi/2} \sin^m x \, dx = \frac{m-1}{m}\int_0^{\pi/2} \sin^{m-2} x \, dx. \quad . \quad . \quad . \quad (4)$$

We shall have to consider separately the four cases which arise by m and n being odd and even.

CASE I. Let m and n be even positive integers. Then, applying formula (3) repeatedly till the power of $\cos x$ becomes zero, and after that applying formula (4) repeatedly, we have

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdot \cdots \frac{1}{m+2}\int_0^{\pi/2} \sin^m x \, dx$$

$$=\frac{(n-1)(n-3)(n-5)\ldots 1}{(m+n)(m+n-2)\ldots(m+2)}\cdot\frac{m-1}{m}\cdot\frac{m-3}{m-2}\cdots\frac{1}{2}\int_0^{\pi/2}dx$$

$$=\frac{\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)\cdots\frac{1}{2}\cdot\left(\frac{m-1}{2}\right)\left(\frac{m-3}{2}\right)\cdots\frac{1}{2}}{\left(\frac{m+n}{2}\right)\left(\frac{m+n-2}{2}\right)\ldots 1}\left(\tfrac{1}{2}\pi\right). \quad \ldots (5)$$

The relation (1) gives, if n is a positive even integer,

$$\Gamma\left(\frac{n+1}{2}\right)=\frac{n-1}{2}\Gamma\left(\frac{n-1}{2}\right)=\frac{n-1}{2}\cdot\frac{n-3}{2}\Gamma\left(\frac{n-3}{2}\right)$$

$$=\text{etc.}=\frac{n-1}{2}\cdot\frac{n-3}{2}\cdot\frac{n-5}{2}\cdots\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right)$$

$$=\frac{n-1}{2}\cdot\frac{n-3}{2}\cdot\frac{n-5}{2}\cdots\frac{1}{2}\cdot\sqrt{\pi}.$$

Hence, by (5),

$$\int_0^{\pi/2}\sin^m x\cos^n x\,dx=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}. \qquad (6)$$

Case II. Let n be an even positive integer and m an odd positive integer. Proceeding as in Case I, we find

$$\int_0^{\pi/2}\sin^m x\cos^n x\,dx=\frac{(n-1)(n-3)(n-5)\ldots 1}{(m+n)(m+n-2)\ldots(m+2)}$$

$$\times\frac{m-1}{m}\cdot\frac{m-3}{m-2}\cdots\frac{2}{3}\int_0^{\pi/2}\sin x\,dx$$

$$=\frac{\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)\cdots\frac{1}{2}\cdot\left(\frac{m-1}{2}\right)\left(\frac{m-3}{2}\right)\ldots 1}{\left(\frac{m+n}{2}\right)\left(\frac{m+n-2}{2}\right)\cdots\frac{3}{2}}\cdot 1$$

$$=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}.$$

Case III. n odd, m even. The integral can be transformed into the one considered in Case II by writing $x+\frac{1}{2}\pi$ for x. Hence the formula (6) is valid in the present case also.

Case IV. n and m both odd positive integers.

Proceeding as in Case I, we have

$$\int_0^{\pi/2}\sin^m x\cos^n x\,dx=\frac{n-1}{m+1}\cdot\frac{n-3}{m+n-2}\cdots\frac{2}{m+3}\times\int_0^{\pi/2}\sin^m x\cos x\,dx.$$

$$=\frac{(n-1)(n-3)(n-5)\ldots 2}{(m+n)(m+n-2)\ldots(m+3)}\cdot\frac{m-1}{m+1}\cdot\frac{m-3}{m-1}\cdots\frac{2}{4}\times\int_0^{\pi/2}\sin x\cos x\,dx.$$

$$=\frac{\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\ldots 1\cdot\left(\frac{m-1}{2}\right)\left(\frac{m-3}{2}\right)\ldots 1}{\left(\frac{m+n}{2}\right)\left(\frac{m+n-2}{2}\right)\ldots 2}\cdot\frac{1}{2}$$

$$=\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}.$$

It is easy to see that the formula (6) is true also when m or n is zero. Hence in every case

$$\int_0^{\pi/2}\sin^m x\cos^n x\,dx=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}.$$

Alternative Formula. The value of the integral under consideration can also be written down from the formula

$$\int_0^{\pi/2}\mathbf{\sin^m x\cos^n x\,dx}$$
$$=\frac{\mathbf{(m-1)(m-3)(m-5)\ldots\times(n-1)(n-3)\ldots}}{\mathbf{(m+n)(m+n-2)(m+n-4)\ldots}}\times\mathbf{k},$$

where the last factor in a product like $m(m-2)\ldots$ is 1 if the other factors are odd, but is 2 if the other factors are even; and k is unity except if m and n are both even, when $k=\frac{1}{2}\pi$. If m or n is zero, then also this formula holds, provided we omit all negative factors in the numerator, and regard 0 as an even number in determining the value of k.

4·14. Trigonometrical transformation. It is possible to break up products of powers of sines and cosines into a sum by trigonometry, and thus integrate such powers easily.

Ex. 1. Integrate $\sin^2 x \cos^4 x$.

Let $\cos x+i\sin x=z$; then $\cos x-i\sin x=z^{-1}$.

Therefore $2\cos x=z+z^{-1}$, $2i\sin x=z-z^{-1}$.

Also by De Moivre's Theorem, $2\cos px=z^p+z^{-p}$.

$$2i\sin px=z^p-z^{-p}.$$

Therefore
$$\begin{aligned}2^2 2^4 i^2 \sin^2 x\cos^4 x&=(z-z^{-1})^2(z+z^{-1})^4\\&=(z^6+z^{-6})+2(z^4+z^{-4})-(z^2+z^{-2})-4\\&=2\cos 6x+2^2\cos 4x-2\cos 2x-4.\end{aligned}$$

Hence

$$\int \sin^2 x\cos^4 x\,dx=-2^{-5}(\tfrac{1}{6}\sin 6x+\tfrac{1}{2}\sin 4x-\tfrac{1}{2}\sin 2x-2x).$$

Ex. 2. Integrate $\sin mx\cos nx$.

Since $\sin mx\cos nx=\frac{1}{2}\{\sin(m+n)x+\sin(m-n)x\}$, we have

$$\int \sin mx\cos nx\,dx=-\frac{\cos(m+n)x}{2(m+n)}-\frac{\cos(m-n)x}{2(m-n)}.$$

4·15. Substitution. Various integrals can be reduced to the forms considered in this chapter by suitable substitutions. Some substitutions of this nature have already been given (§ 3·9).

Ex. Evaluate $\int_0^a x^2(a^2-x^2)^{3/2}\,dx.$

Put $x=a\sin\theta,\ dx=a\cos\theta\,d\theta.$

Then $\theta=0$ when $x=0$, and $\theta=\frac{1}{2}\pi$ when $x=a$.

Hence $\int_0^a x^2(a^2-x^2)^{3/2}\,dx=a^6\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta$

$$=a^6\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{2\Gamma(4)}=a^6\frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2.3.2.1}=\frac{1}{32}\pi a^6.$$

EXAMPLES

Integrate

1. $\sin^5 x.$
2. $\cos^7 x.$
3. $\cos^2 x\sin^3 x.$
4. $\cos^{3/4} x\sin^5 x.$
5. $\operatorname{cosec}^{2/3} x\cos^3 x.$
6. $1/\sin\theta\cos^3\theta.$
7. $\sec x\tan^3 x.$
8. $1/\sqrt{(\cos^3 x\sin^5 x)}.$
9. $\sin^2 x.$
10. $\sin^6 x.$ [*Baroda*, 1957]
11. $\sin^2 x\cos^6 x.$
12. $\sin^4 x\cos^2 x.$

Evaluate

13. $\int_0^{\pi/4}\sin^5\theta\cos^2\theta\,d\theta.$ [*Agra*, 1959]
14. $\int_0^{\pi/4}\sin^4\theta\,d\theta.$ [*Aligarh*, 1956]
15. $\int_0^{\pi/2}\sin^6 x\,dx.$ [*Ban.*,'61]
16. $\int\cos^4 x\,dx.$ [*Travan.*, '41]
17. $\int_0^{\pi/2}\sin^4 x\cos^2 x\,dx.$ [*Baroda*, 1960]
18. $\int_0^{\pi/2}\sin^5 x\cos^8 x\,dx.$
19. Show that $\int_0^1 x^2(1-x^2)^{3/2}\,dx=\pi/32.$ [*Agra*, 1958]
20. Evaluate $\int_0^a x^4\sqrt{(a^2-x^2)}\,dx.$ [*Allahabad*, 1956]

21. Show that $\int_0^a \frac{x^4}{\sqrt{(a^2-x^2)}}\,dx=\frac{3a^4\pi}{16}$. [*Allanabad*, 1958]

22. Evaluate $\int_0^{2a} x^{9/2}(2a-x)^{-1/2}\,dx$. [*Rajputana*, 1954]

23. Prove that $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx=3\pi/128$. [*Alld.*, '59]

24. Evaluate $\int_0^{\pi/8}\cos^3 4x\,dx$. [*Banaras, Geophysics*, '56]

25. If I_n denotes $\int_0^a (a^2-x^2)^n\,dx$, and $n>0$, prove that

$$I_n=\frac{2na^2}{2n+1}I_{n-1}.$$ [*Gorakhpur*, 1960]

26. Evaluate $\int \frac{x^2\,dx}{(4+x^2)^{5/2}}$.

27. Evaluate $\int_0^{\pi/2}\frac{\sin^2\theta\,d\theta}{(1+\cos\theta)^2}$.

28. Evaluate $\int \cos mx\cos nx\,dx$.

29. Evaluate $\int \cos x\cos 2x\cos 3x\,dx$.

Imp 30. If m and n are integers, show that

$\int_0^\pi \sin mx\sin nx\,dx=0$ if $m\neq n$, and $=\frac{1}{2}\pi$ if $m=n$. [*Mad.*, '50]

Imp 31. If m and n are integers, prove that

$$\int_0^\pi \cos mx\sin nx\,dx=\frac{2n}{n^2-m^2} \text{ or } 0$$

according as $n-m$ is odd or even. [*Ujjain*, 1960]

4·2. Integration of $\tan^n x$ and $\cot^n x$. $\tan^n x$ and $\cot^n x$ are also integrated by successive reduction. Thus

$$\int \tan^n x\,dx=\int \tan^{n-2}x\tan^2 x\,dx$$

$$=\int \tan^{n-2} x\,(\sec^2 x-1)\,dx$$

$$=\int \tan^{n-2} x \sec^2 x\,dx-\int \tan^{n-2} x\,dx.$$

Since $\int \tan^{n-2} x \sec^2 x\,dx=(\tan^{n-1} x)/(n-1)$, as is easy to see by putting $\tan x=t$, we get the reduction formula

$$\int \tan^n x\,dx=\frac{\tan^{n-1} x}{n-1}-\int \tan^{n-2} x\,dx.$$

Similarly, or by putting $x+\frac{1}{2}\pi$ for x in the above,

$$\int \cot^n x\,dx=-\frac{\cot^{n-1} x}{n-1}-\int \cot^{n-2} x\,dx.$$

4·21. Integration of $\sec^n$ x and cosec^n x. We can derive a reduction formula as follows:

$$\begin{aligned}\int \sec^n x\,dx &=\int \sec^{n-2} x\,.\,\sec^2 x\,dx\\ &=\sec^{n-2} x\tan x\\ &\qquad -(n-2)\int \sec^{n-3} x\,.\,\sec x\tan x\,.\,\tan x\,dx\\ &=\sec^{n-2} x\tan x-(n-2)\int \sec^{n-2} x\,(\sec^2 x-1)\,dx\\ &=\sec^{n-2} x\tan x+(n-2)\int \sec^{n-2} x\,dx\\ &\qquad -(n-2)\int \sec^n x\,dx.\end{aligned}$$

Transposing the last term to the left and dividing by $n-1$, we get

$$\int \sec^n x\,dx=\frac{\sec^{n-2} x\tan x}{n-1}+\frac{n-2}{n-1}\int \sec^{n-2} x\,dx,$$

which is the required reduction formula.

Similarly, or by putting $x+\frac{1}{2}\pi$ for x in the above, we get

$$\int \operatorname{cosec}^n x\,dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x\,dx.$$

If n is a positive integer, $\int \sec^n x\,dx$ and $\int \operatorname{cosec}^n x\,dx$ can be completely evaluated by repeatedly using the above formulae.

Examples

Integrate

1. $\tan^3 x$. [*Delhi*, 1956]
2. $\tan^4 x$. [*East Panjab*, '51]
3. $\cot^4 x$.
4. $\cot^5 x$.

Evaluate

5. $\displaystyle\int_0^{\pi/4} \tan^5\theta\,d\theta$. [*Gor.*, '60]
6. $\displaystyle\int \frac{d\theta}{\sin^3\theta}$.
7. $\displaystyle\int \frac{d\theta}{\sin^4 \frac{1}{2}\theta}$.
8. $\displaystyle\int_0^{\pi/4} \sec^3 x\,dx$. [*Alld.*, '58]
9. $\displaystyle\int (1+x^2)^{3/2}\,dx$.
10. $\displaystyle\int_0^a (a^2+x^2)^{5/2}\,dx$. [*Bar.*, '59]

4·3. Integration of $1/(a+b\cos x)$.

$$\int \frac{dx}{a+b\cos x} = \int \frac{dx}{a(\cos^2\frac{1}{2}x+\sin^2\frac{1}{2}x)+b(\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x)}$$

$$= \int \frac{dx}{(a+b)\cos^2\frac{1}{2}x+(a-b)\sin^2\frac{1}{2}x}$$

$$= \int \frac{\sec^2\frac{1}{2}x\,dx}{(a+b)+(a-b)\tan^2\frac{1}{2}x}$$

$$= 2\int \frac{dt}{(a+b)+(a-b)t^2}, \text{ where } \tan\tfrac{1}{2}x = t,$$

$\therefore \frac{1}{2}\sec^2 x/2 = dt$

$$= \frac{2}{a-b}\int \frac{dt}{\{(a+b)/(a-b)\}+t^2}.$$

Case I. $(a+b)/(a-b)$ positive. In this case

$$\int \frac{dx}{a+b\cos x}=\frac{2}{a-b}\cdot\frac{\sqrt{(a-b)}}{\sqrt{(a+b)}}\tan^{-1}\left\{\frac{\sqrt{(a-b)}}{\sqrt{(a+b)}}\,t\right\}$$

$$=\frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left\{\frac{\sqrt{(a^2-b^2)}}{(a+b)}\tan\tfrac{1}{2}x\right\}.$$

Case II. $(a+b)/(a-b)$ negative. In this case the integral

$$=\frac{2}{a-b}\cdot\frac{\sqrt{(b-a)}}{2\sqrt{(b+a)}}\log\frac{t-\sqrt{\{(b+a)/(b-a)\}}}{t+\sqrt{\{(b+a)/(b-a)\}}}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}}\log\frac{\sqrt{(b^2-a^2)}\tan\frac{1}{2}x+b+a}{\sqrt{(b^2-a^2)}\tan\frac{1}{2}x-(b+a)}.$$

The value of the integral of $1/(a+b\sin x+c\cos x)$ can be easily deduced from the above, for if we put $b=\beta\sin\theta$, and $c=\beta\cos\theta$, then $a+b\sin x+c\cos x$ assumes the form

$$a+\beta\cos(x-\theta),$$

where β and θ are constants.

Putting now $x-\theta$ equal to, say, t, the integral assumes the form already considered.

Ex. Evaluate $\displaystyle\int_0^{\pi/2}\frac{dx}{4+5\cos x}$. [*Calcutta*, 1960]

$$\int\frac{dx}{4+5\cos x}=\int\frac{dx}{(4+5)\cos^2\frac{1}{2}x+(4-5)\sin^2\frac{1}{2}x}$$

$$=2\int\frac{\frac{1}{2}\sec^2\frac{1}{2}x\,dx}{9-\tan^2\frac{1}{2}x}=2\int\frac{dt}{9-t^2},\text{ where } t=\tan\tfrac{1}{2}x$$

$$=-\tfrac{2}{6}\log\frac{t-3}{t+3}=\tfrac{1}{3}\log\frac{\tan\frac{1}{2}x+3}{\tan\frac{1}{2}x-3}.$$

Therefore $$\int_0^{\pi/2}\frac{dx}{4+5\cos x}=\tfrac{1}{3}\left\{\log\frac{1+3}{1-3}-\log\frac{3}{-3}\right\}$$

$$=\tfrac{1}{3}\log_e 2.$$

4·31. Integration of $1/(a+b\sin x)$. The value of

$$\int \frac{dx}{a+b\sin x}$$

may be easily deduced from that of $\int dx/(a+b\cos x)$ by writing $x-\frac{1}{2}\pi$ for x in the latter.

We may also obtain it independently. Thus

$$\int \frac{dx}{a+b\sin x} = \int \frac{dx}{a(\cos^2 \frac{1}{2}x+\sin^2 \frac{1}{2}x)+2b\sin\frac{1}{2}x\cos\frac{1}{2}x}$$

$$= \int \frac{\sec^2 \frac{1}{2}x\,dx}{a+2b\tan\frac{1}{2}x+a\tan^2\frac{1}{2}x}$$

$$= \frac{2}{a}\int \frac{dt}{t^2+2(b/a)t+1}, \text{ where } t=\tan\tfrac{1}{2}x,$$

$$= \frac{2}{a}\int \frac{dt}{(t+b/a)^2+1-b^2/a^2} = \text{etc.},$$

there being two cases, according as $b<a$ or $>a$.

4·32. Integration of any rational function of $\sin x$ and $\cos x$. If $t=\tan\frac{1}{2}x$ then

$$\sin x = \frac{2\sin\frac{1}{2}x\cos\frac{1}{2}x}{\cos^2\frac{1}{2}x+\sin^2\frac{1}{2}x} = \frac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x}{\cos^2\frac{1}{2}x+\sin^2\frac{1}{2}x} = \frac{1-\tan^2\frac{1}{2}x}{1+\tan^2\frac{1}{2}x} = \frac{1-t^2}{1+t^2}.$$

Also $$\frac{dx}{dt} = \frac{d}{dt}(2\tan^{-1}t) = \frac{2}{1+t^2}.$$

It is evident, therefore, that if the given integrand be a rational function of $\sin x$ and $\cos x$, the new integrand, after substituting $\tan\frac{1}{2}x=t$ will be a rational function of t, and so can be evaluated by breaking it up into partial fractions.

This method is not very convenient in practice, because the function of t thus obtained is a fraction whose denominator

is generally of a high degree in t. Very often it is possible to devise some other method. Thus, by dividing by a suitable power of $\sin x$ or $\cos x$ it may be possible to convert the given integrand into the product of $\sec^2 x$ and a rational function of $\tan x$, or of $\operatorname{cosec}^2 x$ and a rational function of $\cot x$. It is easy after this to evaluate the integral by putting $\tan x$ or $\cot x$ equal to a new variable.

Or, it may be possible to put $\sin x$ or $\cos x$ equal to a new variable and simplify the integral. Also two terms may often be combined into one by changing the constants. Thus $a \sin x + b \cos x$ may often be profitably combined into one term $r \cos(x-\alpha)$ by writing $r \sin\alpha$ for a and $r \cos\alpha$ for b. On the other hand, in some cases the integral may be broken up into two or more parts each of which is easily integrable. For example, we can write

$$\frac{l+m\sin x+n\cos x}{a+b\sin x+c\cos x}$$

as

$$\frac{A(b\cos x-c\sin x)}{a+b\sin x+c\cos x}+\frac{B}{a+b\sin x+c\cos x}+C,$$

where the constants A, B, C can be determined by converting the second expression into a fraction with the denominator $a+b\sin x+c\cos x$ and comparing its numerator with $l+m\sin x+n\cos x$.

Of the three terms in the new expression, the first and the last can be integrated at once, and the middle term can be integrated by putting $b = r\sin\alpha$, $c = r\cos\alpha$.

Ex. Integrate $\quad 1/(a\sin^2\theta + b\cos^2\theta)^2$.

Let $b/a = c$. Then

$$\int\frac{d\theta}{(a\sin^2\theta+b\cos^2\theta)^2}=\int\frac{\sec^4\theta\,d\theta}{(b+a\tan^2\theta)^2}$$

$$=\int\frac{(1+t^2)\,dt}{(b+at^2)^2},\ \text{where } t=\tan\theta,\ =\frac{1}{a}\int\frac{b+at^2+a-b}{(b+at^2)^2}\,dt$$

$$=\frac{1}{a^2}\int\frac{dt}{c+t^2}+\frac{a-b}{a^3}\int\frac{dt}{(c+t^2)^2}$$

$$=\left\{\frac{1}{a^2}+\frac{a-b}{2a^3c}\right\}\int\frac{dt}{c+t^2}+\frac{(a-b)t}{2a^3c(c+t^2)},\ \text{by § 2·7, =etc.}$$

EXAMPLES

Integrate

1. $1/(5+4\cos x)$. [*Panjab*, '57] 2. $1/(5+4\sin x)$.

3. Show that $\displaystyle\int_0^{\pi/2}\frac{d\theta}{1+2\cos\theta}=\frac{1}{\sqrt{3}}\log(2+\sqrt{3})$. [*Jam.*, 1956]

4. Prove that $\displaystyle\int_0^{\pi}\frac{d\theta}{5+3\cos\theta}=\tfrac{1}{4}\pi$. [*Ban.*, *Geoph.*, '57]

5. Prove that $\displaystyle\int_0^{\alpha}\frac{d\theta}{\cos\alpha+\cos\theta}=\operatorname{cosec}\alpha\log\sec\alpha$. [*Nagpur*, 1959]

6. Prove that $\displaystyle\int_0^{\pi}\frac{dx}{1-2a\cos x+a^2}=\frac{\pi}{1-a^2}$ or $\dfrac{\pi}{a^2-1}$, according as $a <$ or > 1. [*Rajasthan*, '60]

Evaluate

7. $\displaystyle\int_0^{\pi/2}\frac{dx}{4+5\sin x}$. [*Sagar*, 1960]

8. $\displaystyle\int\frac{dx}{1+\cos^2 x}$. [*Del.*, '60] 9. $\displaystyle\int\frac{dx}{1+3\sin^2 x}$. [*Banaras*, *Geoph.*, 1960]

Integrate

10. $1/(a^2-b^2\cos^2 x)$, $a>b$. [*Rajasthan*, 1959]
11. $\cos x/(a+b\cos x)$. [*Ban.*, *Geoph.*, 1961]
12. $\sin x/\sqrt{(1+\sin x)}$. [*Allahabad*, 1960], 82
13. $1/(2\sin x+\cos x)^2$. [*Utkal*, 1950]
14. $1/(a^2\cos^2 x+b^2\sin^2 x)$. [*Allahabad*, 1956]
15. $1/(a\sin x+b\cos x)$. [*Aligarh*, 1958]
16. $(2\sin x+3\cos x)/(3\sin x+4\cos x)$. [*Ujjain*, 1960]
17. Show that $\displaystyle\int_0^{\pi}\frac{dx}{3+2\sin x+\cos x}=\tfrac{1}{4}\pi$. [*Panjab*, 1949]

Integrate

18. $1/(a+b\tan\theta)\}$. [*Lucknow*, 1950]
19. $1/(\sin x+\sin 2x)$. [*Andhra*, 1960]
20. $1/(3+2\cos x)\sin x$. [*Panjab*, 1954]

21. $(1+\sin x)/\sin x(1+\cos x)$. [*Banaras*, 1959]

22. $\sin x/\sin(x-\alpha)$. [Hint. Put $x-\alpha=t$.]

23. Evaluate $\displaystyle\int \frac{\tan x\,dx}{\sqrt{(a+b\tan^2 x)}}$, $b>a$. [*P.S.C., U.P.*, '55]

24. Integrate $\sin 2x/(a+b\cos x)^2$.

25. Evaluate $\displaystyle\int_0^{\pi} \frac{dx}{(5+4\cos x)^2}$. [*Agra*, 1943]

4·4. Integration of $\mathbf{x^n \sin mx}$ or $\mathbf{x^n \cos mx}$. Integrals of the form

$$\int x^n \sin mx\,dx \quad \text{or} \quad \int x^n \cos mx\,dx$$

can be evaluated by the method of successive reduction. The reduction formula is easily found by integrating by parts twice. Thus

$$\int x^n \sin mx\,dx = -\frac{x^n \cos mx}{m} + \frac{n}{m}\int x^{n-1}\cos mx\,dx$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\left\{\frac{x^{n-1}\sin mx}{m} - \frac{n-1}{m}\int x^{n-2}\sin mx\,dx\right\};$$

i.e., $$\int x^n \sin mx\,dx = -\frac{x^n\cos mx}{m} + \frac{nx^{n-1}\sin mx}{m^2} - \frac{n(n-1)}{m^2}\int x^{n-2}\sin mx\,dx,$$

which is the required reduction formula.

In the end we shall have to evaluate either $\int x \sin mx\,dx$ or $\int \sin mx\,dx$. The latter is immediately integrable and the former can be evaluated by integrating by parts once. The case of $\int x^n \cos mx\,dx$ is similar.

4·41. Integration of $\mathbf{x^n e^{ax} \sin bx}$ or $\mathbf{x^n e^{ax} \cos bx}$. Integrals of the form

$$\int \underset{I}{x^n}\,\underbrace{e^{ax}\sin bx}_{II}\,dx \quad \text{and} \quad \int x^n e^{ax}\cos bx\,dx$$

can be easily evaluated by repeatedly integrating by parts.

$(a^2+b^2)^{-\frac{1}{2}} e^{ax} \sin\left(bx - \tan^{-1}\frac{b}{a}\right)$

Ex. Integrate $x^2e^{3x}\sin 4x$.

Since $\int e^{3x}\sin 4x\,dx=(3^2+4^2)^{-1/2}e^{3x}\sin(4x-\tan^{-1}\frac{4}{3})$,
we get, if $\alpha=\tan^{-1}\frac{4}{3}$,

$$\int x^2e^{3x}\sin 4x\,dx=\frac{x^2e^{3x}\sin(4x-\alpha)}{5}-\tfrac{2}{5}\int xe^{3x}\sin(4x-\alpha)\,dx$$

$$=\frac{x^2e^{3x}\sin(4x-\alpha)}{5}-\tfrac{2}{5}\left\{\frac{xe^{3x}\sin(4x-2\alpha)}{5}-\tfrac{1}{5}\int e^{3x}\sin(4x-2\alpha)\,dx\right\}$$

$$=\tfrac{1}{125}e^{3x}\{25x^2\sin(4x-\alpha)-10x\sin(4x-2\alpha)+2\sin(4x-3\alpha)\}.$$

4·42. Integration of $e^{ax}\sin^n bx$ or $e^{ax}\cos^n bx$.

The integrals

$$\int e^{ax}\sin^n bx\,dx \quad\text{and}\quad \int e^{ax}\cos^n bx\,dx$$

can be evaluated by transforming $\sin^n bx$ or $\cos^n bx$ in to a sum of sines or cosines of multiples of x (§4·14) or by successive reduction. The reduction formula can be obtained by integrating by parts twice.

Thus

$$\int e^{ax}\sin^n bx\,dx=\frac{\sin^n bx\,.\,e^{ax}}{a}-\frac{nb}{a}\int\sin^{n-1}bx\cos bx\,e^{ax}\,dx$$

$$=\frac{\sin^n bx\,e^{ax}}{a}-\frac{nb}{a}\left[\frac{\sin^{n-1}bx\cos bx\,e^{ax}}{a}-\frac{b}{a}\int\{(n-1)\sin^{n-2}bx\cos^2 bx-\sin^n bx\}e^{ax}\,dx\right].$$

Now the integral on the right can be written as

$$\int\{(n-1)\sin^{n-2}bx\,(1-\sin^2 bx)-\sin^n bx\}e^{ax}\,dx,$$

i.e., $$(n-1)\int\sin^{n-2}bx\,e^{ax}\,dx-n\int\sin^n bx\,e^{ax}\,dx.$$

Transposing the second of these two integrals, multiplied by its proper coefficient, to the left and dividing by $1+n^2b^2/a^2$, we get

$$\int e^{ax}\sin^n bx\,dx=\frac{a\sin bx-nb\cos bx}{a^2+n^2b^2}\,e^{ax}\sin^{n-1}bx$$
$$+\frac{n(n-1)}{a^2+n^2b^2}\,b^2\int e^{ax}\sin^{n-2}bx\,dx,$$

which is the required reduction formula.

The case of $\int e^{ax}\cos^n bx\,dx$ is similar.

4·43. Reduction formula for integrals of the type $\int \cos^m x \sin nx\, dx$.

Integrating by parts, we have

$$\int\cos^m x\sin nx\,dx=-\frac{\cos^m x\cos nx}{n}$$
$$-\frac{m}{n}\int\cos^{m-1}x\cos nx\sin x\,dx.$$

Replacing $\cos nx\sin x$ by $\sin nx\cos x-\sin(n-1)x$, and thus breaking up the integral on the right into two, transposing one of them to the left and dividing by a constant, we get the reduction formula

$$\int\cos^m x\sin nx\,dx=-\frac{\cos^m x\cos nx}{m+n}$$
$$+\frac{m}{m+n}\int\cos^{m-1}x\sin(n-1)x\,dx.$$

Integrals involving $\cos^m x\cos nx$, $\sin^m x\cos nx$, and $\sin^m x\sin nx$ can be treated in the same way.

Examples

Integrate

1. $x^2\sin 2x$. 2. $x\sin^2 x$.

3. Prove that $\int_0^\pi \theta\sin^2\theta\cos\theta\,d\theta=-\frac{4}{9}$. [*Andhra*, '60]

4. Evaluate $\int_0^a \sqrt{(a^2-x^2)}\{\cos^{-1}(x/a)\}^2\,dx.$

5. If $$u_n=\int_0^{\pi/2} x^n \sin x\,dx,$$
and $n>1$, prove that
$$u_n+n(n-1)u_{n-2}=n(\pi/2)^{n-1}.$$ [*Baroda*, 1960]

Hence evaluate $\int_0^{\pi/2} x^5 \sin x\,dx.$ [*Banaras*, 1962]

Integrate

6. $e^{2x}\cos^3 x.$ 7. $e^{ax}\sin^3 x.$

8. $e^x(x\cos x+\sin x).$ [*Bombay*, 1940]

9. Evaluate $\int_0^\infty xe^{-2x}\cos x\,dx.$ [*P.S.C., U.P., Forest*, '60]

10. Integrating by parts twice, or otherwise, obtain a reduction formula for
$$I_m=\int_0^\infty e^{-x}\sin^m x\,dx,$$
where $m\geqslant 2$, in the form $(1+m^2)I_m=m(m-1)I_{m-2}$; and hence evaluate I_4. [*Allahabad*, 1962], Alld 80.

11. If $I(m, n)=\int_0^{\pi/2}\cos^m x\cos nx\,dx$, prove that
$$I(m, n)=\{m(m-1)/(m^2-n^2)\}I(m-2, n).$$ [*Allahabad*, 1960]

12. Prove that
$$\int_0^{\pi/2}\cos^m x\sin nx\,dx$$
$$=\frac{1}{m+n}+\frac{m}{m+n}\int_0^{\pi/2}\cos^{m-1}x\sin(n-1)x\,dx.$$
[*Nagpur*, 1959]

13. Prove that
$$\int_0^{\pi/2}\cos^{n-2}x\sin nx\,dx=\frac{1}{n-1}\ (n>1 \text{ and integral}).$$ [*Alld.*, '60], 82

14. Prove that, if n is a positive integer,
$$\int_0^{\pi/2}\cos^n x\cos nx\,dx=\frac{\pi}{2^{n+1}}.$$ [*Delhi*, 1958]

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4·5. Other transcendental functions. There are no general propositions which will enable us to integrate every function of $\sin x$, $\cos x$, etc., or of e^x, $\log x$, etc. The methods of Chapter I should be tried. Very often some suitable substitution will reduce a given function to some easily integrable form. In particular, an integral of a rational function of e^x is transformed into an integral of a rational function of t by the substitution $e^x=t$.

In some cases a function of x may be expanded in powers of x and the result integrated term by term. The student must, however, remember that this process is not always justifiable: but the consideration of the conditions under which this can be done is beyond the scope of the present volume.

Ex. 1. Evaluate $\int_0^\infty e^{-x}x^n\,dx$, n being a positive integer. [*Rajasthan*, 1959]

Integrating by parts,

$$\int_0^\infty e^{-x}x^n\,dx=\Big[-x^n e^{-x}\Big]_0^\infty+n\int_0^\infty e^{-x}x^{n-1}\,dx.$$

$$\text{Now } \lim_{x\to\infty} x^n e^{-x}=\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}$$

$$=\text{etc.}=\lim_{x\to\infty}\frac{n(n-1)\ldots1}{e^x}=0.$$

Hence $\qquad \Big[-x^n e^{-x}\Big]_0^\infty=0.$

Therefore $\quad \int_0^\infty e^{-x}x^n\,dx=n\int_0^\infty e^{-x}x^{n-1}\,dx.$

Applying this reduction formula repeatedly, and remembering that $\Big[-e^{-x}\Big]_0^\infty=1$, we get

$$\int_0^\infty e^{-x}x^n\,dx=n!$$

Amburg Tripathi Examples.

Ex. 2. Integrate $1/(1+e^x-2e^{2x})$.

Putting $e^x=t$, we have

$$\int\frac{dx}{1+e^x-2e^{2x}}=\int\frac{dt}{t(1+t-2t^2)}=\text{etc.}$$

Ex. 3. Find a reduction formula for $\int x^n(\log x)^m\,dx$.
[*Agra*, 1957]

Integrating by parts,

$$\int x^n(\log x)^m\,dx=\frac{x^{n+1}(\log x)^m}{n+1}-\frac{m}{n+1}\int x^n(\log x)^{m-1}\,dx,$$

which is the required reduction formula.

Ex. 4. Evaluate $\int_0^1\frac{1}{x}\log\frac{1+x}{1-x}\,dx$.

The integrand

$$=(1/x)\,.\,2(x+x^3/3+x^5/5+\ldots)$$
$$=2(1+x^2/3+x^4/5+\ldots).$$

Hence the integral

$$=2\left[x+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\ldots\right]_0^1$$
$$=2\left\{1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right\}=\pi^2/4.$$

EXAMPLES

Integrate

1. $1/(e^x-1)(e^x+3)$.
2. $1/(1+e^x)(1+e^{-x})$.
3. $1/(e^x-1)^2$. [*Sagar*, '50]
4. $e^{m\cos^{-1}x}$.
5. $e^{m\tan^{-1}x}/(1+x^2)^2$.
6. $e^x(1+x)/(2+x)^2$.
7. $e^x(x^2+1)/(1+x)^2$.
8. $e^x(x^2+3x+3)/(x+2)^2$.
9. If I_n denotes $\int z^n e^{1/z}\,dz$, show that
$(n+1)!\,I_n=I_0+e^{1/z}(1!\,z^2+2!\,z^3+\ldots+n!\,z^{n+1})$.

Integrate

10. $\log(1+x^2)$. [*Ban.*, '58] 11. $\log x \sin^{-1} x$. [*Ban.*, '60]

12. $\log\{x+\sqrt{(x^2-a^2)}\}$. 13. $\{\log(x+1)\}/x^2$.

14. $e^x(x\log x+1)/x$. [*Patna*, 1950]

15. $\sec x \log(\sec x+\tan x)$. [*Delhi, Hons.*, 1960]

Evaluate

16. $\int_0^1 x^2 \sin^{-1} x\, dx$. [*Patna*, '40] 17. $\int_0^1 \log(1/x-1)\, dx$.

18. $\int_0^\infty \dfrac{x\, dx}{1+e^x}$. [*Agra*, '43] 19. $\int x^4(\log x)^2\, dx$.

20. Evaluate $\int x^2 \log(1-x^2)\, dx$, [*Rajasthan*, '60]

and deduce that

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$$\frac{1}{1.5}+\frac{1}{2.7}+\frac{1}{3.9}+\ldots=\tfrac{8}{9}-\tfrac{2}{3}\log_e 2.$$

Integrate by expanding the integrand

21. e^{-x^2}. 22. $\sin mx/x$.

23. $\sin(1/x)$. 24. $\sqrt{(\cos x)}$.

Integrate

25. $\sin^{-1}\sqrt{\{x/(a+x)\}}$. [*Agra*, 1958]

26. $\sqrt{(e^{2x}+ae^x)}$. [*Bombay*, 1940]

27. $1/\cosh^3 x$. (Put $\cosh x=\sec y$.) [*Utkal*, 1949]

28. $\cos x \cosh x$.

29. $(\cosh x+\sinh x \sin x)/(1+\cos x)$.

30. Find a reduction formula for

$$\int \tanh^n x\, dx.$$

31. Prove that

$$\int \frac{dx}{a+b\cosh x} = \frac{2}{\sqrt{(b^2-a^2)}} \tan^{-1}\left\{\frac{\sqrt{(b-a)}}{\sqrt{(b+a)}} \tanh \tfrac{1}{2}x\right\},$$

$$\text{or } \frac{2}{\sqrt{(a^2-b^2)}} \tanh^{-1}\left\{\frac{\sqrt{(a-b)}}{\sqrt{(a+b)}} \tanh \tfrac{1}{2}x\right\},$$

according as $b>a$ or $<a$.

[Hint. Remember that $\cosh^2\theta - \sinh^2\theta = 1$, $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$; so the method of § 4·3 can be followed.]

Integrate

32. $(\tan^{-1} x)/(1+x)^2$.

33. $\sin \log x/x^3$. [*U.P.E.S.*, '54]

34. $\cos^{-1} x/x^3$.

35. $(\sin^{-1} x)/(1-x^2)^{3/2}$. [*Patna*, 1950]

36. $(\sin^{-1} x)^2$.

37. $\sqrt{(\sec x - 1)}$. [*Raj.*, '57]

38. $\tan^{-1}\sqrt{x}$.

39. $\cos 2x \log(1+\tan x)$.

40. $\sec x \operatorname{cosec} x/\log \tan x$. [*Delhi, Hons.*, 1960]

41. $1/\sqrt{\{\sin^3 x \sin(x+\alpha)\}}$. [*Aligarh*, 1958]

[Hint. Integrand $= \operatorname{cosec}^2 x\, dx/\sqrt{\{\cos\alpha + \sin\alpha \cot x\}}$. Put $\cot x = t$.]

42. $\sqrt{\left\{\dfrac{\sin(x-\alpha)}{\sin(x+\alpha)}\right\}}$. [*Allahabad*, 1962]

43. $1/\cos\theta\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta + c^2)}$.

44. Evaluate $\displaystyle\int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2}\,dx$. [*Nagpur*, 1956]

45. Prove that $\displaystyle\int_0^1 \frac{x^3 \sin^{-1} x\, dx}{(1-x^2)^{1/2}} = \frac{7}{9}$. [*Madras*, 1951]

Examples on Chapter IV

Integrate

1. $\sin^5 x \cos^2 x$.

2. $\sin^2 x \cos^3 x$.

3. $\sin^{3/2} x \cos^3 x$.

4. $1/\sin^4 x \cos^2 x$. [*Alld.*, '51]

Evaluate

5. $\int_0^{\pi/2} \sin^5 x \cos^6 x \, dx.$ 6. $\int_0^{\pi/6} \cos^4 3\phi \sin^2 6\phi \, d\phi.$

7. $\int_0^1 x^4 (1-x^2)^{5/2} \, dx.$ [*Gorakhpur*, 1960]

8. $\int_0^1 x^{3/2} \sqrt{(1-x)} \, dx.$ [*Baroda*, 1956]

9. $\int_0^a \frac{x^4 \, dx}{(a^2+x^2)^4}.$ [*Gor.*, '60] 10. $\int_0^\infty \frac{dx}{(a^2+x^2)^{n+1/2}}.$

11. $\int_0^a x^3 (2ax-x^2)^{3/2} \, dx.$ [*Allahabad*, 1950]

12. If $$\phi(n) = \int_0^{\pi/4} \tan^n x \, dx,$$
show that $$\phi(n) + \phi(n-2) = 1/(n-1),$$ [*Lucknow*, 1962]
and deduce the value of $\phi(5)$. [*Gorakhpur*, 1960]

13. Show that $\int_0^a \frac{dx}{x+\sqrt{(a^2-x^2)}} = \frac{1}{4}\pi.$ [*Baroda*, 1960]

Evaluate

14. $\int \frac{dx}{(1+2\cos x)^3}.$ [*Lucknow*, 1945]

15. $\int x^3 \tan x \, dx.$ [*Panjab*, 1937] [Hint. Expand $\tan x$.]

16. $\int \frac{dx}{\sin(x-a)\sin(x-b)}.$ [*Banaras*, 1960]

[Hint. Integrand $= \{\cot(x-a) - \cot(x-b)\} \operatorname{cosec}(a-b)$.]

17. $$\int \frac{3+4\sin x+2\cos x}{3+2\sin x+\cos x} dx.$$

18. $\int \frac{a+b\cos\theta+c\sin\theta}{1+\sin\theta} d\theta.$ 19 $\int \frac{p\sin x+q\cos x}{a\sin x+b\cos x} dx.$

20. Find the value of $\int_0^{\pi/4} \frac{\sin 2\theta \, d\theta}{\sin^4\theta+\cos^4\theta}$ [*Baroda*, 1960] A.U. 71.

21. Use the substitution L.U. 70

$$\tan \tfrac{1}{2}\theta=\sqrt{\{(1+e)/(1-e)\}}\tan \tfrac{1}{2}u$$

to evaluate $\int \frac{d\theta}{(1+e\cos\theta)^2}$.

22. If $U_n=\int_0^{\pi/2} \theta \sin^n\theta \, d\theta$ and $n>1$, prove that

$$U_n=\{(n-1)/n\}U_{n-2}+1/n^2.$$

Deduce that $U_5=\frac{149}{225}$. [*Gorakhpur*, 1959]

23. If $I_n=\int_0^{\pi/2} x^n \sin(2p+1)x \, dx$, prove that

$$I_n+\frac{n(n-1)}{(2p+1)^2}I_{n-2}=(-1)^p\frac{n}{(2p+1)^2}\left(\frac{\pi}{2}\right)^{n-1},$$

n and p being positive integers. [*Madras*, 1942]

Evaluate

24. $\int_0^{\pi/2} x^3 \sin 3x \, dx.$ [*Nagpur*, 1950]

25. $\int \frac{x^2 \, dx}{(x\sin x+\cos x)^2}.$ [*Poona*, 1956]

[Write numerator as $x\cos x \,.\, x\sec x$ and integrate by parts, taking $x\sec x$ as first function.]

26. $\int_0^{\infty} e^{-\alpha x}\cos\beta x\cos\gamma x \, dx$, where $\alpha>0$.

27. Prove that $\int_0^1 \frac{\sqrt{(1-x^2)}}{1-x^2\sin^2\alpha}dx=\frac{\pi}{4\cos^2\frac{1}{2}\alpha}.$ [*Vikram*, '61]

28. Evaluate $\int e^x \frac{1+\sin x}{1+\cos x}dx.$ [*Allahabad*, 1960]

29. Find the reduction formula for I_{mn}, where

$$I_{mn}=\int_0^{\pi/2}(\cos x)^m \sin nx \, dx.$$

Deduce that $I_{mm}=\dfrac{1}{2^{m+1}}\left\{2+\dfrac{2^2}{2}+\dfrac{2^3}{3}+\ldots+\dfrac{2^m}{m}\right\}.$

30. If $S_n=\displaystyle\int_0^{\pi/2}\frac{\sin(2n-1)x}{\sin x}\,dx,\quad V_n=\int_0^{\pi/2}\left(\frac{\sin nx}{\sin x}\right)^2 dx,$
(n an integer), show that

$$S_{n+1}-S_n=0,\quad V_{n+1}-V_n=S_{n+1}.$$

31. If m and n are positive integers and

$$f(m, n)=\int_0^1 x^{n-1}(\log x)^m\,dx,$$

prove that $\quad f(m, n)=-(m/n)f(m-1, n).$

Deduce that $f(m, n)=(-1)^m\,m!/n^{m+1}.$ [*Vikram*, 1961]

32. Integrate $1/x\,l(x)\,l^2(x)\,l^3(x)\ldots l^r(x)$,
where $l^r(x)$ means $\log\log\log\ldots x$, the log being repeated r times.

33. Show that $\displaystyle\int_0^1\frac{\log x}{1+x}\,dx=-\int_0^1\frac{\log(1+x)}{x}\,dx=-\pi^2/12.$

34. If n is an integer greater than 1, prove that

$$\int_0^\infty\frac{dx}{\{x+\sqrt{(1+x^2)}\}^n}=\frac{n}{n^2-1}.$$

CHAPTER V

DEFINITE INTEGRALS

5·1. Definitions. As already defined,

$$\int_a^b f(x)\,dx$$

means $F(b)-F(a)$, where $F(x)=\int f(x)\,dx$, i.e., where $dF(x)/dx=f(x)$.

$\int_a^b f(x)\,dx$ is called the *definite integral* of $f(x)$ from a to b or between the *limits* a and b.

a and b are called its *lower* and *upper* limits. The interval (a, b) is called the *range of integration.*

To distinguish it from a definite integral, the function $F(x)$, i.e., $\int f(x)\,dx$, is sometimes called the indefinite integral of $f(x)$. It should be noticed that an indefinite integral can be written, if necessary, as a definite integral. For

$$\int_a^x f(x)\,dx$$

is equal to $F(x)-F(a)$, and, therefore, is identical with the indefinite integral of $f(x)$, viz. $F(x)+C$, if $C=-F(a)$.

5·2. General properties of the definite integral.

Let

$$\int f(x)\,dx=F(x), \text{ so that } \int_a^b f(x)\,dx=F(b)-F(a).$$

Then

(i) $$\int_a^b \mathbf{f(x)\,dx}=\int_a^b \mathbf{f(t)\,dt}.$$

For both sides are equal to $F(b)-F(a)$.

(ii) $$\int_a^b \mathbf{f(x)\,dx}=-\int_b^a \mathbf{f(x)\,dx}.$$

For $F(b)-F(a)=-\{F(a)-F(b)\}$.

(iii) $$\int_a^b \mathbf{f(x)\,dx}=\int_a^c \mathbf{f(x)\,dx}+\int_c^b \mathbf{f(x)\,dx}.$$

For the right-hand side is equal to

$$F(c)-F(a)+F(b)-F(c),$$

which is equal to $F(b)-F(a)$.

We can generalise this into the following:

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \dots + \int_{c_{r-1}}^{c_r} f(x)\,dx + \int_{c_r}^{b} f(x)\,dx.$$

For the right-hand side is equal to $F(c_1) - F(a) + F(c_2) - F(c_1) + F(c_3) - F(c_2) + \dots + F(c_r) - F(c_{r-1}) + F(b) - F(c_r)$, which is equal to $F(b) - F(a)$.

(iv) $$\boxed{\int_0^a \mathbf{f(x)\,dx} = \int_0^a \mathbf{f(a-x)\,dx.}}$$ ✓

For, putting $a - x = t$, the righ-hand side becomes equal to

$$-\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt = \int_0^a f(x)\,dx.$$

(v) $$\boxed{\int_{-a}^a \mathbf{f(x)\,dx} = 0 \quad \textbf{or} \quad 2\int_0^a \mathbf{f(x)\,dx,}}$$

according as $f(x)$ is an odd or an even function of x.

For $$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx. \quad (1)$$

Now $$\int_{-a}^0 f(x)\,dx = -\int_a^0 f(-t)\,dt, \text{ where } t = -x,$$

$$= \int_0^a f(-t)\,dt, \text{ by (ii)}, = \int_0^a f(-x)\,dx, \text{ by (i)},$$

$$= -\int_0^a f(x)\,dx \text{ if } f(x) \text{ is an odd function of } x,$$

or $$+\int_0^a f(x)\,dx \text{ if } f(x) \text{ is an even function of } x.$$

Substituting in (1) we get the result at once.

(vi) $\displaystyle\int_0^{2a} \mathbf{f(x)\,dx} = 2\int_0^a \mathbf{f(x)\,dx}$ if $f(2a-x)=f(x)$,

and $\qquad\qquad = 0 \qquad$ if $f(2a-x)=-f(x)$.

For $\displaystyle\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

$$= \int_0^a f(x)\,dx - \int_a^0 f(2a-y)\,dy, \text{ where } x=2a-y,$$

$$= \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = 2\int_0^a f(x)\,dx \text{ or } 0,$$

according as $f(2a-x)$ is equal to $f(x)$ or to $-f(x)$.

In particular, $\displaystyle\int_0^{\pi} \mathbf{f(\sin x)\,dx} = 2\int_0^{\pi/2} \mathbf{f(\sin x)\,dx}.$

Also $\displaystyle\int_0^{\pi} \boldsymbol{\phi}(\mathbf{\cos x})\,\mathbf{dx} = \mathbf{0 \text{ or } 2}\int_0^{\pi/2} \boldsymbol{\phi}(\mathbf{\cos x})\,\mathbf{dx},$

according as $\phi(z)$ is an odd or an even function of z.

5·3. Evaluation of definite integrals. In many cases it is possible to evaluate a definite integral by special methods, although it may not be easy or even possible to find the corresponding indefinite integral.

Ex. 1. Evaluate $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx.$ [*Roorkee*, 1962]

Let $\displaystyle I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx.$

Then $\displaystyle I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$, by (iv), § 5·2.

Adding the two values of I, we get

$$2I = \int_0^{\pi} \frac{(\pi-x+x)\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$$

$$= -\pi\Big[\tan^{-1}\cos x\Big]_0^{\pi} = -\pi(-\tfrac{1}{4}\pi - \tfrac{1}{4}\pi).$$

Hence $\qquad I = \tfrac{1}{4}\pi^2.$

Ex. 2. Evaluate $\int_0^{\pi} \sin^4 x \, dx$.

$$\int_0^{\pi} \sin^4 x \, dx = 2\int_0^{\pi/2} \sin^4 x \, dx, \text{ by (vi), § 5·2,}$$

$$= \frac{2\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{2\Gamma(3)} = \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\pi}{2\,.\,1} = 3\pi/8.$$

Ex. 3. Evaluate $\int_0^{\pi/2} \log \sin x \, dx$. [*Banaras*, 1962]

Let $$I = \int_0^{\pi/2} \log \sin x \, dx.$$

Then $I = \int_0^{\pi/2} \log \cos x \, dx$, by (iv), § 5·2.

Adding the two value of I, we get

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x) \, dx$$

$$= \int_0^{\pi/2} \{\log(2 \sin x \cos x) - \log 2\} \, dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \tfrac{1}{2}\pi \log 2$$

$$= \tfrac{1}{2}\int_0^{\pi} \log \sin u \, du - \tfrac{1}{2}\pi \log 2, \text{ where } x = \tfrac{1}{2}u.$$

Now $\frac{1}{2}\int_0^{\pi} \log \sin u \, du = \int_0^{\pi/2} \log \sin u \, du$, by (vi), § 5·2, $= I$.

Substituting this in the value of $2I$ obtained above, we get

$$2I = I - \tfrac{1}{2}\pi \log 2.$$

Hence $$\int_0^{\pi/2} \log \sin x \, dx = -\tfrac{1}{2}\pi \log_e 2.$$

N.B. This result is required in solving several of the examples which follow, and should be committed to memory.

Examples

Evaluate

1. $\int_0^{\pi} \sin^3 x \, dx$. 2. $\int_0^{\pi} \cos^6 x \, dx$.

3. $\int_0^{\pi} \sin^3 \theta (1 + 2\cos\theta)(1 + \cos\theta)^2 \, d\theta$. [*Nagpur*, 1953]

4. $\displaystyle\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$. [*Bar.*, '60] 5. $\displaystyle\int_0^{\pi} \theta \sin^3 \theta\, d\theta$.

Show that

6. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{(\sin x)}\,dx}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}=\tfrac{1}{4}\pi$. [*Ban.*, *Geoph.*, 1961]

7. $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\,dx=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$. [*Alld.*, '60]

8. $\displaystyle\int_0^{\pi} \frac{x\tan x\,dx}{\sec x+\tan x}=\pi(\tfrac{1}{2}\pi-1)$. [*P.S.C.*, *U.P.*, 1960] 76

9. Show that $\displaystyle\int_0^{\pi/2} \log\tan x\,dx=0$. [*Banaras*, 1951]

10. Apply the substitution $x=\pi-y$ to the integral

$$\int_0^{\pi} x\sin^6 x\cos^4 x\,dx,$$

and hence obtain its value. [*Gorakhpur*, 1960]

11. Find the value of $\displaystyle\int_0^1 \frac{\sin^{-1} x}{x}\,dx$. [*Agra*, 1960]

12. Show that $\displaystyle\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta=\tfrac{1}{8}\pi\log_e 2$.

[*Agra*, 1962]

13. Show that $\displaystyle\int_0^{\pi} \log(1+\cos x)\,dx=\pi\log_e \tfrac{1}{2}$. [*Alld.*, '60]

14. Evaluate $\displaystyle\int_0^{\infty} \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}$. [*Rajasthan*, 1961]

[Hint. Put $x=\tan\theta$.]

15. Show that $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx=\tfrac{1}{8}\pi\log_e 2$. [*Raj.*, '60]

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5·4. The integral as the limit of a sum. We have so far looked upon integration as the operation which is the inverse of differentiation, and we defined the integral of a function $f(x)$ as the function which when differentiated will give us $f(x)$.

But it is also possible to regard the definite integral, and hence also the indefinite integral (see § 5·1), as the limit of the sum of a finite series of numbers, when the number of terms of the series tends to infinity whilst each term of the series tends to zero. Thus we can *define*

$$\int_a^b f(x)\,dx$$

by the equation

$$\int_a^b \mathbf{f(x)\,dx} = \mathbf{lim}_{n\to\infty}\,\mathbf{h[f(a)+f(a+h)+f(a+2h)} \\ \mathbf{+\ldots+f\{a+(n-1)h\}]},$$

where
$$\mathbf{b-a=nh},$$

and establish the equivalence of the two definitions in what is known as the Fundamental Theorem of the Integral Calculus, which asserts that the operations of differentiation and of integration (as now defined) are inverse operations.* The truth of this theorem is obvious from § 1·7.

The problem of integration in various problems of geometry and other branches of knowledge generally presents itself in the form of a summation. (See for example § 1·8, in which the determination of areas was considered.) Historically also this method of regarding integration is of prior origin. The sign of integration ∫ is merely the old form of *s*, the initial letter of the word sum.

For the purposes of modern rigorous mathematics too the viewpoint which regards the integral as the limit of a certain sum is of the greatest importance. We say that a function is *integrable* if this limit exists, even though we may not be able to express it in terms of the known functions. The object of modern theories of integration is so to modify the above definition as to include more and more complicated functions in the class of integrable functions.

*It is supposed that the function which is integrated is continuous.

The new definition of the Integral provides an answer to the question raised in § 1·6, but it will be more convenient to investigate this point later. See § 6·11 (iii).

5·41. Integration from definition as the limit of a sum. In some elementary cases it is possible to find the value of an integral direct from the sum-definition. The process is naturally tedious, but is instructive. Before the invention of the calculus, a similar procedure had to be applied in every case when an area or a volume was wanted.

Ex. Evaluate $\int_a^b x^2\,dx$ directly from the definition of the integral as the limit of a sum. [*Lucknow*, 1962]

$$\int_a^b x^2\,dx = \lim_{n\to\infty} h[a^2+(a+h)^2+(a+2h)^2+\dots$$
$$+\{a+(n-1)h\}^2], \text{ where } b-a=nh,$$
$$=\lim_{n\to\infty} h[na^2+\{1+2+3+\dots+(n-1)\}2ah$$
$$+\{1^2+2^2+\dots+(n-1)^2\}h^2]$$
$$=\lim_{n\to\infty} h[na^2+\tfrac{1}{2}n(n-1)2ah+\tfrac{1}{6}(n-1)(2n-1)\}nh^2]$$
$$=\lim_{n\to\infty}[nh\,.\,a^2+nh\,.\,(n-1)h\,.\,a$$
$$+\tfrac{1}{3}\,.\,(n-1)h\,.\,(n-\tfrac{1}{2})h\,.\,nh]$$
$$=(b-a)a^2+(b-a)^2a+\tfrac{1}{3}(b-a)^3$$
$$=\tfrac{1}{3}(b-a)\{3a^2+3(b-a)a+b^2-2ab+a^2\}$$
$$=\tfrac{1}{3}(b-a)(a^2+ab+b^2)=\tfrac{1}{3}b^3-\tfrac{1}{3}a^3.$$

5·42. Summation of series. The definition of the integral as the limit of a sum enables us to express the limits of sums of series of a certain type as definite integrals and thus to evaluate them.

The value of the required limit can be written down by the formula of § 5·4, viz.,

$$\lim_{n\to\infty}\sum_{r=0}^{r=n-1}\{f(a+rh)\}h=\int_a^b f(x)\,dx,$$

where $nh=b-a$,

or, more conveniently, by the formula derived from the above by putting $a=0$ and $b=1$, viz.,

$$\mathbf{lim}_{n\to\infty}\sum_{r=0}^{r=n-1}\left\{\mathbf{f}\left(\frac{\mathbf{r}}{\mathbf{n}}\right)\right\}\frac{\mathbf{1}}{\mathbf{n}}=\int_0^1 \mathbf{f}(\mathbf{x})\,\mathbf{dx}.$$

In order that a series may be capable of being summed by this formula, it must possess the following properties :

(i) It must be possible to write the terms in the form $\frac{1}{n}f\left(\frac{r}{n}\right)$, so that $1/n$, which tends to zero, is a factor of every term, and, apart from this factor, all the terms are the same function of r/n, which varies in value from term to term in arithmetical progression with the common difference $1/n$.

(ii) The number of terms should be n; but since each term tends to zero, the addition or omission of one or two terms (or any finite number of terms) will not alter the required limit; that is, even

$$\lim_{n\to\infty}\sum_{r=k}^{r=n+l} f\left(\frac{r}{n}\right)\frac{1}{n}=\int_0^1 f(x)\,dx,$$

provided k and l are independent of n.

An easy way to write down the definite integral corresponding to a given series is to write the latter as $\Sigma\{f(r/n)\}(1/n)$, and therefore the required limit as

$$\lim_{n\to\infty}\sum\left\{f\left(\frac{r}{n}\right)\right\}\frac{1}{n}.$$

To write down the corresponding definite integral, replace r/n by x, its common difference, viz., $1/n$, by dx, and $\lim_{n\to\infty}\Sigma$ by $\int$; and insert the values of r/n for the first and the last terms (or the limits of such values) as the lower and upper limits respectively in the integral.

This procedure will evidently give us $\int f(x)\,dx$ taken between the proper limits.

NOTE. It can be shown (by putting cn equal to a new variable n') that the rule given above in italics is applicable even when the number of terms in the given series is $cn+$ a constant instead of $n+$ a constant. In this case the upper limit of the integral will come out as c instead of 1.

Ex. 1. Determine by integration the limit to which the sum

$$\frac{n^{1/2}}{n^{3/2}}+\frac{n^{1/2}}{(n+3)^{3/2}}+\frac{n^{1/2}}{(n+6)^{3/2}}+\ldots+\frac{n^{1/2}}{\{n+3(n-1)\}^{3/2}}$$

tends as n is indefinitely increased. [*Baroda*, 1959]

The $(r+1)$th term is $\dfrac{n^{1/2}}{(n+3r)^{3/2}}$, i.e., $\dfrac{1/n}{(1+3r/n)^{3/2}}$.

We, therefore, require the value of

$$\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1/n}{(1+3r/n)^{3/2}}.$$

By the rule given above this is equal to $\displaystyle\int_0^1\frac{dx}{(1+3x)^{3/2}}$

$$=\left[-\frac{2}{3(1+3x)^{1/2}}\right]_0^1=-\tfrac{1}{3}+\tfrac{2}{3}=\tfrac{1}{3}.$$

Ex. 2. Find the limit, when n tends to infinity, of the product $(1+1/n)(1+2/n)^{1/2}(1+3/n)^{1/3}\ldots(1+n/n)^{1/n}$. [*Lucknow*, 1958]

Let the required limit be A. Then

$$\log A=\lim_{n\to\infty}\{\log(1+1/n)+\tfrac{1}{2}\log(1+2/n)+\ldots+(1/n)\log(1+n/n)\}$$

$$=\lim_{n\to\infty}\Sigma(1/r)\log(1+r/n)$$

$$=\lim_{n\to\infty}\sum\left\{\frac{1}{r/n}\log\left(1+\frac{r}{n}\right)\right\}\frac{1}{n}=\int_0^1\frac{1}{x}\log(1+x)\,dx$$

$$=\int_0^1\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots\right)dx=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

$$=\pi^2/12.$$

Therefore $$A=e^{\pi^2/12}.$$

Examples

From the definition of a definite integral as the limit of a sum evaluate

1. $\int_1^2 x\,dx.$ [*Baroda*, '60]
2. $\int_a^b e^x\,dx.$ [*Banaras*, '62]
3. $\int_a^b \cos x\,dx.$
4. $\int_a^b \sin\theta\,d\theta.$ [*Poona*, '56]
5. $\int_a^b \frac{1}{\sqrt{x}}\,dx.$
6. $\int_a^b \frac{1}{x^2}\,dx.$ [*Cal.*, '43]

Find the limit, when $n\to\infty$, of the series

7. $(1/n^3)(1+4+9+16+\ldots+n^2).$ [*Lucknow*, 1945]
8. $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}.$ [*Allahabad*, 1959]
9. $\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots+\frac{1}{8n}.$ [*Allahabad*, 1962]
10. $\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+(n+1)^2}.$ [*Delhi*, 1960]
11. $\frac{1}{n}\left\{\sin^{2k}\frac{\pi}{2n}+\sin^{2k}\frac{2\pi}{2n}+\sin^{2k}\frac{3\pi}{2n}+\ldots+\sin^{2k}\frac{\pi}{2}\right\}.$
12. Evaluate $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{(n^2-r^2)}}.$ [*Rajasthan*, '62]
13. Find the limit, when n tends to infinity, of the series

$$\frac{1}{1+n^3}+\frac{4}{8+n^3}+\frac{9}{27+n^3}+\ldots+\frac{r^2}{r^3+n^3}+\ldots+\frac{1}{2n}.$$

[*Allahabad*, 1960]

14. Show that the limit of the sum

$$\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\dots+\frac{1}{3n},$$

when n is indefinitely increased, is $\log_e 3$. [*Lucknow*, 1955]

[Here the number of terms is $2n+1$; so consider it as the sum of two series, $1/n+\dots+1/2n$ and $1/(2n+1)+\dots+1/3n$, and evaluate the limit of each separately.]

15. Evaluate $\lim_{n\to\infty}\left\{\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}\right.$

$$\left.+\frac{3}{n^2}\sec^2\frac{9}{n^2}+\dots+\frac{1}{n}\sec^2 1\right\}. \qquad [\textit{Gujrat}, \text{'57}]$$

16. Find the limit, as n tends to infinity, of the product

$$\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\dots\left(1+\frac{n}{n}\right)\right\}^{1/n}. \quad [\textit{Panjab}, \text{'56}]$$

17. Prove that

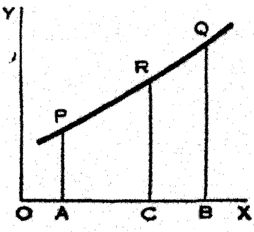
$$\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\dots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$$

is equal to $2e^{(\pi-4)/2}$. [*Delhi, Honours*, 1960]

18. Apply the definition of a definite integral as the limit of a sum to evaluate $\lim_{n\to\infty}(n!/n^n)^{1/n}$. [*P.S.C., U.P.*, '57]

5·5. Geometrical meaning. (i) We have already seen that

$$\int_a^b f(x)\,dx$$



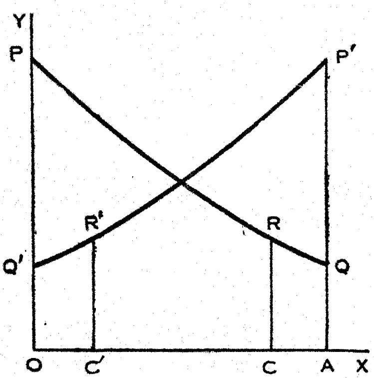
can be interpreted as the area $ABQRP$, where PRQ is the curve $y=f(x)$, and PA, QB are the ordinates at $x=a$ and $x=b$.

If the abscissa of R is c, the formula

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$$

of § 5·2 merely expresses the fact that

the area $ABQP$ = the area $ACRP$ + the area $CBQR$.



(ii) To interpret the formula

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx,$$

consider the curves

$$y=f(x), \quad . \quad . \quad (1)$$

and $y=f(a-x)$. . . (2)

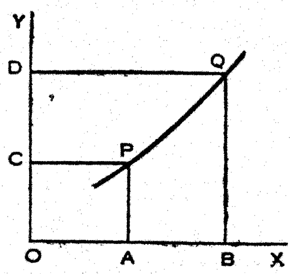
Let these curves be PQ and $Q'P'$. Let P and Q' be on the axis of y, and P' and Q on $x=a$. Let A be the foot of the ordinate of P' or Q.

The ordinate $R'C'$ at $x=x_1$ in the second curve is, by (2), equal to $f(a-x_1)$ and so is the same as the ordinate RC at $x=a-x_1$ in the first curve. Hence the area $OAQP$ can be made to coincide with the area $AOQ'P'$ by applying the former to the latter in such a way that the corner O of the former falls on the corner A of the latter, and the corner A of the former falls on the corner O of the latter. So these areas must be equal. The formula in question merely expresses this fact.

(iii) The formula of integration by parts also becomes obvious when we look into its geometrical significance.

Let PQ be the curve whose parametrical equation is

$$x=\phi(t),\ y=\psi(t).$$



Let PA, QB be the ordinates of P and Q, and PC, QD the perpendiculars drawn from P and Q to the y-axis.

Let P and Q correspond to the values a and b of t.

Then the area $ABQP$

$$=\int_{OA}^{OB} y\,dx = \int_{t=a}^{t=b} y\frac{dx}{dt}\,dt$$

$$=\int_a^b \psi(t)\phi'(t)\,dt.$$

Again, the area $CDQP = \int_{\mathrm{OC}}^{\mathrm{OD}} x\,dy = \int_{t=a}^{t=b} x \frac{dy}{dt}\,dt$

$$= \int_a^b \phi(t)\,\psi'(t)\,dt.$$

The area of the rectangle $OBQD = \phi(b)\psi(b)$, and the area of the rectangle $OAPC = \phi(a)\psi(a)$.

Hence the formula for integration by parts, viz.,

$$\int_a^b \psi(t)\phi'(t)\,dt = \Big[\psi(t)\phi(t)\Big]_a^b - \int_a^b \psi'(t)\phi(t)\,dt,$$

simply expresses the fact that

area $ABQP$ = [rect. $OBQD$ − rect. $OAPC$] − area $CDQP$.

5·6. Improper integrals. Let $f(x)$ be continuous for all values of x from a to b $(b > a)$, except that $f(x) \to \infty$ as $x \to b$; then we define

$$\int_a^b f(x)\,dx \text{ to mean } \lim_{\epsilon \to 0} \int_a^{b-\epsilon} f(x)\,dx,$$

provided that the limit is a definite number.

Similarly, if $f(x) \to \infty$ as $x \to a$ and $f(x)$ is otherwise continuous, we define

$$\int_a^b f(x)\,dx \text{ to mean } \lim_{\epsilon \to 0} \int_{a+\epsilon}^{b} f(x)\,dx,$$

provided that the limit is a definite number.

Again, if $f(x)$ is continuous for all values of x from a to b, except that $f(x) \to \infty$ as $x \to c$, where c lies between a and b, we define

$$\int_a^b f(x)\,dx \text{ to mean}$$

$$\lim_{\epsilon \to 0} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon' \to 0} \int_{c+\epsilon'}^{b} f(x)\,dx,$$

provided that each limit is a definite number.

In some cases each limit may not separately be a definite number, but

$$\lim_{\epsilon\to 0}\left\{\int_a^{c-\epsilon} f(x)\,dx+\int_{c+\epsilon}^{b} f(x)\,dx\right\}$$

may exist and be a definite number, say A. Then A is called the principal value of

$$\int_a^b f(x)\,dx.$$

The above definitions hold also when the limit of $f(x)$ at $x=a$, b, or c is $-\infty$, or when the limits on the right and on the left at $x=c$ are infinite, but of different signs.

Ex. 1. $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x)}}=\lim_{\epsilon\to 0}\int_0^{1-\epsilon}\frac{dx}{\sqrt{(1-x)}}$

$$=\lim_{\epsilon\to 0}\Big[-2\sqrt{(1-x)}\Big]_0^{1-\epsilon}=\lim_{\epsilon\to 0}[-2\sqrt{\epsilon}+2]=2.$$

Ex. 2. $\displaystyle\int_0^1 \frac{dx}{1-x}=\lim_{\epsilon\to 0}\int_0^{1-\epsilon}\frac{dx}{1-x}$

$$=\lim_{\epsilon\to 0}\Big[-\log(1-x)\Big]_0^{1-\epsilon}=\lim_{\epsilon\to 0}[-\log\epsilon+0].$$

Since $\lim_{\epsilon\to 0}\log\epsilon$ is $-\infty$, $\displaystyle\int_0^1\frac{dx}{1-x}$ is meaningless.

Ex. 3. $\displaystyle\int_0^1\frac{dx}{\sqrt{x}}=\lim_{\epsilon\to 0}\int_\epsilon^1\frac{dx}{\sqrt{x}}=\lim_{\epsilon\to 0}\Big[2\sqrt{x}\Big]_\epsilon^1$

$$=\lim_{\epsilon\to 0}(2-2\sqrt{\epsilon})=2.$$

Ex. 4. $\displaystyle\int_0^1\frac{dx}{x^3}=\lim_{\epsilon\to 0}\int_\epsilon^1\frac{dx}{x^3}$

$$=\lim_{\epsilon\to 0}\Big[-\frac{1}{2x^2}\Big]_\epsilon^1=\lim_{\epsilon\to 0}\Big[-\tfrac{1}{2}+\frac{1}{2\epsilon^2}\Big].$$

Since $\lim_{\epsilon\to 0}(1/\epsilon^2)=\infty$, $\displaystyle\int_0^1\frac{dx}{x^3}$ is meaningless.

Ex. 5. $$\int_{-1}^{1}\frac{dx}{x^{2/3}}=\lim_{\epsilon\to 0}\int_{-1}^{-\epsilon}\frac{dx}{x^{2/3}}+\lim_{\epsilon'\to 0}\int_{\epsilon'}^{1}\frac{dx}{x^{2/3}}$$
$$=\lim_{\epsilon\to 0}\Big[3x^{1/3}\Big]_{-1}^{-\epsilon}+\lim_{\epsilon'\to 0}\Big[3x^{1/3}\Big]_{\epsilon'}^{1}$$
$$=\lim\,[-3\epsilon^{1/3}+3]+\lim_{\epsilon'\to 0}[3-3(\epsilon')^{1/3}]$$
$$=6.$$

Ex. 6. $$\int_{-1}^{1}\frac{dx}{x^2}=\lim_{\epsilon\to 0}\int_{-1}^{-\epsilon}\frac{dx}{x^2}+\lim_{\epsilon'\to 0}\int_{\epsilon'}^{1}\frac{dx}{x^2}$$
$$=\lim_{\epsilon\to 0}\Big[-\frac{1}{x}\Big]_{-1}^{-\epsilon}+\lim_{\epsilon'\to 0}\Big[-\frac{1}{x}\Big]_{\epsilon'}^{1}$$
$$=\lim_{\epsilon\to 0}\Big[\frac{1}{\epsilon}-1\Big]+\lim_{\epsilon'\to 0}\Big[-1+\frac{1}{\epsilon'}\Big].$$

The integral $\int_{-1}^{1}\frac{dx}{x^2}$ has, therefore, no meaning. The principal value also does not exist, for

$$\lim_{\epsilon\to 0}(1/\epsilon-1-1+1/\epsilon)$$

does not exist.

Ex. 7. $$\int_{-1}^{2}\frac{dx}{x^3}=\lim_{\epsilon\to 0}\Big[-\frac{1}{2x^2}\Big]_{-1}^{-\epsilon}+\lim_{\epsilon'\to 0}\Big[-\frac{1}{2x^2}\Big]_{\epsilon'}^{2}$$

and so the integral has no meaning. But the principal value

$$=\lim_{\epsilon\to 0}\Big\{-\frac{1}{2\epsilon^2}+\tfrac{1}{2}-\tfrac{1}{8}+\frac{1}{2\epsilon^2}\Big\}=\tfrac{3}{8},$$

and thus the principal value exists.

5·7. Some theorems about definite integrals. In what follows $f(x)$ is supposed to be a continuous function of x.

(i) If $f(x)\geqslant 0$ for all values of x such that $a\leqslant x\leqslant b$, then

$$\int_a^b f(x)\,dx\geqslant 0.$$

For, under the given conditions, the sum whose limit is equal to the definite integral cannot be negative.

This theorem often enables us to say whether the result of integration should be positive or negative, and so serves as a check.

(ii) If U be the greatest value (upper bound) and L the least value (lower bound) of $f(x)$ in the interval (a, b) then

$$(b-a)L \leqslant \int_a^b f(x)\,dx \leqslant (b-a)\,U.$$

This follows at once if we apply the previous theorem to the integrals $\int_a^b \{f(x)-L\}\,dx$ and $\int_a^b \{U-f(x)\}\,dx$.

(iii)
$$\int_a^b f(x)\,dx=(b-a)f(\xi),$$
where ξ is some number such that $a \leqslant \xi \leqslant b$.

This follows from (ii), because $\int_a^b f(x)\,dx$ which lies between $(b-a)L$ and $(b-a)U$ must be equal to $(b-a)A$, where A lies between L and U. Now, since $f(x)$ is continuous, there must be a value ξ of x between a and b such that $f(\xi)=A$.

(iv) If $g(x)$ is positive and L and U are defined as before, then

$$L\int_a^b g(x)dx \leqslant \int_a^b f(x)\,g(x)\,dx \leqslant U\int_a^b g(x)\,dx.$$

This follows at once by applying theorem (i) to the integrals

$$\int_a^b \{f(x)-L\}\,g(x)\,dx \quad\text{and}\quad \int_a^b \{U-f(x)\}\,g(x)\,dx.$$

(v) If $f(x)$ is a continuous function of x in the interval (a, b) and $g(x)$ is a continuous function of x which is $\geqslant 0$ then

$$\int_a^b f(x)\,g(x)\,dx=f(\xi)\int_a^b g(x)\,dx,$$

where $a \leqslant \xi \leqslant b$.

This follows from (iv), because $\int_a^b f(x)\,g(x)\,dx$ which lies between $L\int_a^b g(x)\,dx$ and $U\int_a^b g(x)\,dx$ must be equal to

$B\int_a^b g(x)\,dx$, where B lies between L and U. Now, since $f(x)$ is continuous, there must be a value ξ of x between a and b such that $f(\xi)=B$.

This theorem is known as the *first Mean Value Theorem of the Integral Calculus.* Theorem (iii) is a particular case of it obtained by putting $g(x)=1$.

For ξ we can also write $a+\theta(b-a)$, where $0\leqslant\theta\leqslant 1$.

Examples on Chapter V

1. Prove that $\int_0^{2a}\phi(x)\,dx=\int_0^a\{\phi(x)+\phi(2a-x)\}\,dx$,

and illustrate the theorem geometrically. [*Gorakhpur*, 1959]

2. If $\phi(x)=\phi(2a-x)$, show that

$$\int_0^{2a}\phi(x)\,dx=2\int_0^a\phi(x)\,dx,\quad \text{[Rajasthan, 1960]}$$

and evaluate
$$\int_0^{\pi}\cos^{2n}x\,dx.\quad \text{[Allahabad, 1956]}$$

3. If $f(x+mp)=f(x)$ for all integral values of m, prove that

$$\int_0^{np}f(x)\,dx=n\int_0^p f(x)\,dx,$$

where n is a positive or negative integer. [*Allahabad*, 1962]

Give a geometrical interpretation.

4. Prove that $\int_0^{\pi/2}\phi(\sin 2x)\sin x\,dx=\int_0^{\pi/2}\phi(\sin 2x)\cos x\,dx$

$$=\sqrt{2}\int_0^{\pi/4}\phi(\cos 2x)\cos x\,dx.$$

5. If m and n are positive and m is an integer, prove that

$$\int_0^1 x^{n-1}(1-x)^{m-1}\,dx=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$=\frac{1.2.3.\ldots(m-1)}{n(n+1)(n+2)\ldots(n+m-1)}.\quad \text{[Luck., '45]}$$

6. Show that $\int_0^{\pi} \frac{x\,dx}{a^2-\cos^2 x}=\frac{\pi^2}{2a\sqrt{(a^2-1)}}$, $a>1$.
[*Gorakhpur*, 1959]

7. Evaluate $\int_0^{\pi/2} \frac{dx}{1+\cot x}$. [*Baroda*, 1959]

8. Show that $\int_0^{\pi} \frac{x^2 \sin 2x \sin(\frac{1}{2}\pi \cos x)}{2x-\pi}\,dx=\frac{8}{\pi}$.
[*Allahabad*, 1962]

[Hint. Put $x-\frac{1}{2}\pi=t$; then apply § 5·2 (v).]

9. Evaluate $\int_0^{\pi/2} x \cot x\,dx$.

10. Prove that $\int_0^{\pi/2} x^2 \operatorname{cosec}^2 x\,dx=\pi \log 2$. [*Banaras*, '57]

11. Prove that, if $0 \leqslant \alpha \leqslant 1$ and the positive value of the square root is taken,

$$\int_{-1}^{1} \frac{dx}{\sqrt{\{1-2\alpha x+\alpha^2\}}}=2.$$

What is the value of the integral if $\alpha>1$? [*P.S.C., U.P.*, 1955]

12. Prove that $\int_0^{\pi} \frac{\sin n\theta}{\sin \theta}\,d\theta$ is equal to 0 or π according as n is an even or odd positive integer. [*Delhi*, 1960]

By means of a reduction formula or otherwise, prove that

$$\int_0^{\pi} \frac{\sin^2 n\theta}{\sin^2 \theta}\,d\theta=n\pi,$$

where n is a positive integer.

13. Find $\int_a^b x^3\,dx$

immediately from its definition as the limit of a sum.
[*Lucknow*, 1950]

14. Show that

$\lim_{n\to\infty}[\{\sqrt{(n+1)}+\sqrt{(n+2)}+\ldots+\sqrt{(2n)}\}/n\sqrt{n}]$
$=\frac{2}{3}(2\sqrt{2}-1)$. [*Aligarh*, 1953]

15. Find the limit, when n tends to infinity, of the series

$$\frac{\sqrt{n}}{(3+4\sqrt{n})^2}+\frac{\sqrt{n}}{\sqrt{2}(3\sqrt{2}+4\sqrt{n})^2}+\frac{\sqrt{n}}{\sqrt{3}(3\sqrt{3}+4\sqrt{n})^2}+\dots+\frac{1}{49n}.$$

[*Lucknow*, 1960]

16. Evaluate $\lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{n}\sqrt{\left(\frac{n+r}{n-r}\right)}$. [*P.S.C., U.P.*, '55]

17. Prove that $I\equiv\int_1^\infty e^{-x^2}\,dx<\int_1^\infty xe^{-x^2}\,dx,$ and hence that $I<1/2e$. [*Bombay*, 1950]

18. Evaluate $\int_0^\pi \frac{\cos nx}{1-2a\cos x+a^2}\,dx$, where n is a positive integer and $a<1$.

[Hint. Use the expansion $(1-a^2)/(1-2a\cos x+a^2)$
$=1+2a\cos x+2a^2\cos 2x+\dots+2a^r\cos rx+\dots$.]

MISCELLANEOUS EXAMPLES

Integrate

1. $(1-x^2)\sqrt{x}$.
2. $1/\{x+\sqrt{(x-1)}\}$. [*Del.*, '45]
3. $\cos^{-1}(1/x)$. [*All.*, '40]
4. $e^{2x}/(e^x-1)$.
5. $xe^x\sin^2 x$. [*Panj.*, '44]
6. $x\tan^2 x$.
7. $[\{\tan(1/x)\}/x]^2$.
8. $\sec^2 x/(\tan^2 x-1)$.
9. $1/\sqrt{\{(1-x^2)\sin^{-1}x\}}$.
10. $e^{2x}\cos(3x+\pi/3)$.
11. $\cos\frac{1}{2}x/\sqrt{(\sin^2\frac{1}{2}x+4)}$.
12. $(x+1)/\sqrt{(4x-x^2)}$. [*Agra*, 1960]
13. $x^2\tan^{-1}x$. [*Andhra*, 1943]
14. $(1+x^2)^{-3/2}e^{m\arctan x}$.
15. $\cot x/\sqrt{(\sin x)}$.
16. $\sec^4 x/\sqrt{(\tan x)}$.
17. $\sqrt{(4+x^2)}/x^6$.
18. $(ax^2+c)^{-3/2}$. [Put $x=1/t$]

19. $(1-x^2)/(1+x^2)\sqrt{(1+x^4)}$. [Put $x+1/x=t$.]

20. $e^x(1-\sin x)/(1-\cos x)$. 21. $(1+\cos x)/\sin x\cos x$.

22. $\cos x/(1+\sin x)(2-\sin x)$.

23. Show that, when $f(x)$ is of the form $a+bx+cx^2$,

$$\int_0^1 f(x)\,dx=\tfrac{1}{6}\{f(0)+4f(\tfrac{1}{2})+f(1)\}.$$

24. Prove that

$$\int uv\,dx=uv_1-u'v_2+u''v_3-u'''v_4+\ldots+(-1)^{n-1}u^{(n-1)}v_n$$
$$+(-1)^n\int u^{(n)}v_n\,dx,$$

where dashes denote differentiation with respect to x and

$$v_r=\int v_{r-1}\,dx.$$

Hence evaluate $\int x^4\sin x\,dx$.

25. Transform the integral $\displaystyle\int\frac{dx}{(a+b\cos x)^2}$

by the substitution $\cos\theta=(a\cos x+b)/(a+b\cos x)$.

Evaluate $\displaystyle\int_0^{\pi/2}\frac{dx}{a+b\cos x}\ (a>b)$

and deduce, or otherwise find, the value of

$$\int_0^{\pi/2}\frac{\cos x}{(a+b\cos x)^2}\,dx. \qquad [U.P.F.S., 1953]$$

26. Obtain a reduction formula for

$$U_n=\int\frac{dx}{(a+b\cos x)^n}$$

in the form $U_n=\dfrac{A\sin x}{(a+b\cos x)^{n-1}}+BU_{n-1}+CU_{n-2}$. [*Poona*, '56]

Find U_2.

27. Prove that

$$\int\frac{dx}{(x-p)\sqrt{\{(x-p)(x-q)\}}}=\frac{2}{q-p}\sqrt{\left(\frac{x-q}{x-p}\right)}.$$

28. Evaluate $\int_\alpha^\beta \frac{dx}{\sqrt{\{(x-\alpha)(\beta-x)\}}}$, $\beta>\alpha$. [*Allahabad*, 1960]

[Hint. Put $x=\alpha\cos^2\theta+\beta\sin^2\theta$.]

29. By using the substitution $x=1/t$, or otherwise, prove that $$\int_{1/3}^1 \frac{(x-x^3)^{1/3}}{x^4}=6.$$

30. Integrate $\operatorname{cosec}^2 x \log\{\cos x+\sqrt{(\cos 2x)}\}$.

31. Show that, if n is a positive integer, then
$$\int_0^{2\pi} \frac{\cos(n-1)x-\cos nx}{1-\cos x}dx=2\pi,$$
and deduce that $\int_0^{2\pi}\left(\frac{\sin \frac{1}{2}nx}{\sin\frac{1}{2}x}\right)^2 dx=2n\pi$.

32. Evaluate the integral
$$\int_1^\infty \frac{x^4+1}{x^2(x^2+1)^2}dx.$$

33. Evaluate $\int_0^1 \frac{dx}{1+2x\cosh\alpha+x^2}$.

34. Show that if p be an integer,
$$\int_0^1 x^{2p-1}\log(1+x)dx=\frac{1}{2p}\left[\frac{1}{1.2}+\frac{1}{3.4}+\dots+\frac{1}{(2p-1)2p}\right].$$

35. Prove that

(i) $\int_0^\infty \frac{x\,dx}{(1+x)(1+x^2)}=\pi/4$;

(ii) $\int_1^\infty \frac{(x^2+3)dx}{x^6(x^2+1)}=\frac{1}{30}(58-15\pi)$.

36. Find the condition that $\int \frac{ax^2+2bx+c}{(Ax^2+2Bx+C)^2}dx$ be rational.

[Find the condition that the terms involving a logarithm or an inverse tangent be absent.]

37. Prove that $\int_x^{1/x} \frac{dx}{1+x^4} = \frac{1}{\sqrt{2}} \cot^{-1} \frac{\sqrt{2}}{1/x - x}$.

38. By using the substitution $x = 2a - t$, show that

$$\int_0^{2a} (2ax - x^2)^{n/2} \operatorname{vers}^{-1} \frac{x}{a}\, dx = \tfrac{1}{2}\pi \int_0^{2a} (2at - t^2)^{n/2}\, dt.$$

Hence evaluate the integral.

39. Evaluate $\int_0^\infty \frac{\log(1+x^2)}{1+x^2}\, dx$. [*Allahabad*, 1962]

40. Show that the sum of the infinite series

$$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \ldots (a>0, b>0)$$

can be expressed in the form $\int_0^1 \frac{t^{a-1}}{1+t^b}\, dt$; and hence prove that

$$1 - \tfrac{1}{4} + \tfrac{1}{7} - \tfrac{1}{10} + \tfrac{1}{13} - \tfrac{1}{16} + \ldots = \tfrac{1}{3}(\pi/\sqrt{3} + \log_e 2).$$ [*Raj.*, '54]

41. Investigate a formula of reduction for

$$\int \frac{x^{2n+1}\, dx}{(1-x^2)^{1/2}},$$

and by means of this integral show that

$$\frac{1}{2n+2} + \frac{1}{2} \cdot \frac{1}{2n+4} + \frac{1.3}{2.4} \cdot \frac{1}{2n+6} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{2n+8} + \cdots$$
$$= \frac{2.4.6 \ldots 2n}{3.5.7 \ldots (2n+1)}.$$ [*Allahabad*, 1958]

Sum also the series

$$\frac{1}{2n+1} + \frac{1}{2} \cdot \frac{1}{2n+3} + \frac{1.3}{2.4} \cdot \frac{1}{2n+5} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{2n+7} + \cdots.$$

[*Allahabad*, 1955]

42. If $I_n = \int x^n \cos \beta x\, dx$ and $J_n = \int x^n \sin \beta x\, dx$, prove

that $\beta I_n = x^n \sin \beta x - n J_{n-1}$;

and $\beta J_n = -x^n \cos \beta x + n I_{n-1}$.

43. If m and n are integers, m is greater than n and $m+n$ is even, prove that

$$\int_0^{\pi/2} \cos^m x \cos nx\, dx = \frac{\pi \,.\, m!}{2^{m+1}\{\frac{1}{2}(m+n)\}!\{\frac{1}{2}(m-n)\}!}.$$

[*P.S.C., U.P.*, 1958]

44. Prove that

$$\int \sin n\theta \sec\theta\, d\theta = \frac{-2\cos(n-1)\theta}{n-1} - \int \sin(n-2)\theta \sec\theta\, d\theta.$$

Hence or otherwise evaluate

$$\int_0^{\pi/2} \frac{\cos 5\theta \sin 3\theta}{\cos\theta} d\theta.$$

45. Prove that

(i) $\int_a^b f(a+b-x)\, dx = \int_a^b f(x)\, dx,$ [*Delhi*, '51]

(ii) $\int_0^{b-c} f(x+c)\, dx = \int_c^b f(x)\, dx.$

46. Prove that $\int_0^{\pi/2} \frac{x\cos x \sin x\, dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi}{4ab^2(a+b)},$

where a and b are positive. [*Poona*, 1957]

47. Prove that $\int_0^{\pi/2} \frac{\cos^2\theta}{\cos^2\theta + 4\sin^2\theta} d\theta = \pi/6.$

48. Evaluate $\int_0^{2\pi} \frac{\sin^2\phi}{a-b\cos\phi} d\phi,\ a>b>0.$

49. Show that $\int_0^{2\pi} \frac{dx}{a+b\cos x+c\sin x} = \frac{2\pi}{\sqrt{(a^2-b^2-c^2)}}$

when $a>\sqrt{(b^2+c^2)}>0.$ [*Allahabad*, 1962]

50. Prove that $\int_0^{\pi} \log(1-2a\cos x+a^2)dx = \pi \log a^2$ if

$a^2>1$, or 0 if $a^2<1$.

51. If m and n are positive integers, prove that

$$\int_0^b (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1} \frac{m!\,n!}{(m+n+1)!}.$$

[*Banaras*, 1957]

[Hint. Put $x=a+(b-a)y$.]

52. If n is a positive integer, prove that

$$\int_0^{\pi/2} \sin^n x \, dx > \int_0^{\pi/2} \sin^{n+1} x \, dx.$$

Deduce that $\frac{1}{2}\pi$ lies between

$$\frac{2.2.4.6.6....2n.2n}{1.3.3.5.5.7....(2n-1)(2n+1)}$$

and

$$\frac{2.2.4.4.6.6....(2n-2).(2n-2).2n}{1.3.3.5.5....(2n-1)(2n-1)}.$$

[This is known as *Wallis's value of* π.] [*I.A.S.*, '52]

53. Prove that, if $e<1$,

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{dx}{\sqrt{(1-e^2\cos^2 x)}} = 1+\frac{1^2}{2^2}e^2+\frac{1^2.3^2}{2^2.4^2}e^4 +\frac{1^2.3^2.5^2}{2^2.4^2.6^2}e^6+\dots.$$

54. Find the limit, when $n\to\infty$, of

$$\frac{n}{(n+1)\sqrt{(2n+1)}}+\frac{n}{(n+2)\sqrt{\{2(2n+2)\}}}+\frac{n}{(n+3)\sqrt{\{3(2n+3)\}}} +\dots+\frac{n}{2n\sqrt{(n.3n)}}.$$

55. Show that the limit, when n is increased indefinitely, of

$$\frac{(n-m)^{1/3}}{n}+\frac{(2^2n-m)^{1/3}}{2n}+\frac{(3^2n-m)^{1/3}}{3n}+\dots+\frac{(n^3-m)^{1/3}}{n^2}$$

is 3/2.

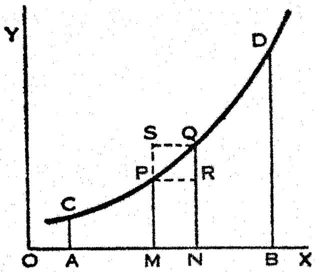
CHAPTER VI

AREAS OF CURVES

6·1. Areas of curves given by Cartesian equations. We have proved before (§ 1·8) that the area bounded by the curve $y=f(x)$, the axis of x, and the ordinates at $x=a$ and $x=b$ is given by

$$\int_a^b y\,dx.$$

The following is an alternative proof.



Let CD be the curve $y=f(x)$, where $f(x)$ is a continuous function of x in the domain (a, b), and suppose*, for the sake of convenience, that y goes on increasing as x increases from a to b. Let CA, DB be the ordinates at $x=a$ and $x=b$.

Let P be any point (x, y) on the curve and let PM be its ordinate.

Then the area $AMPC$ is some function of x, say $\phi(x)$.

Let Q be any other point $(x+h, y+k)$ on the curve, and let QN be its ordinate. Let PR and QS be the perpendiculars from P and Q to NQ and MP produced respectively.

Then the area $ANQC=\phi(x+h)$.

*This restriction can be easily removed as in § 1·8.

Hence

$$\frac{\phi(x+h)-\phi(x)}{h}=\frac{\text{area } ANQC-\text{area } AMPC}{h}$$

$$=\frac{\text{area } MNQP}{h}. \quad . \quad . \quad . \quad (1)$$

Now the area of the rectangle $MNRP=yh$, and that of the rectangle $MNQS=(y+k)h$. Assuming as an axiom that the area $MNQP$ lies in magnitude between the areas of the rectangles $MNRP$ and $MNQS$, it follows from (1) that

$$\frac{\phi(x+h)-\phi(x)}{h}$$

lies between $y+k$ and y. Taking limits, we see that

$$\lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}=y,$$

i.e.,

$$\frac{d\phi(x)}{dx}=f(x). \quad . \quad . \quad . \quad (2)$$

Consequently, if $F(x)$ is any known integral of $f(x)$,

$$\phi(x)=F(x)+C, \quad . \quad . \quad . \quad (3)$$

where C is some constant. To determine C, put $x=a$ in (3). Now $\phi(a)=0$, since $\phi(a)$ is equal to the area $AMPC$ when M coincides with A.

Hence $\qquad F(a)+C=0,$

or $\qquad C=-F(a).$

Consequently $\phi(x)=F(x)-F(a)$.

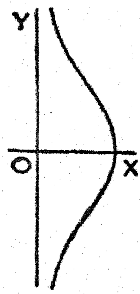
Therefore

the **area ABDC** $=\phi(b)=F(b)-F(a)=\int_a^b \mathbf{f(x)\,dx}$.

Ex. 1. Find the area included between the curve

$$xy^2=4a^2(2a-x)$$

and its asymptote.



The curve is symmetrical about the x-axis and is as shown. It cuts the x-axis at $x=2a$. The asymptote is the y-axis.

Hence the required area $=A$, say,

$$=2\int_0^{2a} y\,dx$$

$$=4a\int_0^{2a} \frac{\sqrt{(2a-x)}}{\sqrt{x}}\,dx.$$

Putting $x=2a\sin^2\theta$ and $dx=4a\sin\theta\cos\theta\,d\theta$, we get

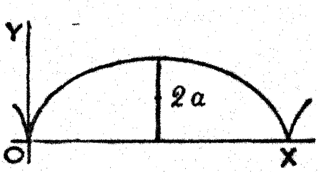
$$A=16a^2\int_0^{\pi/2}\cos^2\theta\,d\theta=16a^2\,\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}{2\Gamma(2)}=4\pi a^2.$$

Ex. 2. Find the area included between the cycloid

$$x=a(\theta-\sin\theta),$$
$$y=a(1-\cos\theta),$$

and its base. [*Allahabad*, 1950]

Since the cycloid is symmetrical with respect to the line $x=a\pi$, and the base is the x-axis, the required area



$$=2\int_0^{a\pi} y\,dx=2\int_{x=0}^{x=a\pi} y\,dx$$

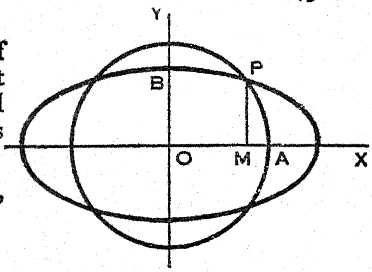
$$=2\int_{\theta=0}^{\theta=\pi} y\cdot\frac{dx}{d\theta}\,d\theta$$

$$=2a^2\int_0^{\pi}(1-\cos\theta)^2\,d\theta$$

$$=8a^2\int_0^{\pi}\sin^4\tfrac{1}{2}\theta\,d\theta=16a^2\int_0^{\pi/2}\sin^4\phi\,d\phi,\text{ where }\tfrac{1}{2}\theta=\phi,$$

$$=16a^2\frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{1}{2})}{2\Gamma(3)}=4a^2\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\pi=3\pi a^2.$$

Ex. 3. Find the area common to the circle $x^2+y^2=4$ and the ellipse $x^2+4y^2=9$.



Let P be the point of intersection, in the first quadrant, of the circle and the ellipse, and construct as in the figure.

The required area, say α,

$$=4\times\text{area } OAPB$$

$$=4(\text{area } OMPB + \text{area } MAP).$$

Solving $\qquad x^2+y^2=4,$

i.e., $\qquad 4x^2+4y^2=16,$

and $\qquad x^2+4y^2=9,$

we see that, for P, $\qquad x=\sqrt{(7/3)}.$

Also, for the ellipse, $\qquad y=\frac{1}{2}\sqrt{(9-x^2)},$

and for the circle, $\qquad y=\sqrt{(4-x^2)}.$

Hence $\alpha=4\int_0^{\sqrt{(7/3)}}\frac{1}{2}\sqrt{(9-x^2)}\,dx+4\int_{\sqrt{(7/3)}}^{2}\sqrt{(4-x^2)}\,dx$

$$=2\left[\tfrac{1}{2}x\sqrt{(9-x^2)}+\tfrac{9}{2}\sin^{-1}\tfrac{1}{3}x\right]_0^{\sqrt{(7/3)}}$$

$$+4\left[\tfrac{1}{2}x\sqrt{(4-x^2)}+2\sin^{-1}\tfrac{1}{2}x\right]_{\sqrt{(7/3)}}^{2}$$

$=$etc.

6·11. Remarks.

(i) The smaller of the two values of x between the ordinates at which the area lies should be chosen as the lower limit.

(ii) If y is negative for all values of x from a to c $(c>a)$, then, as is evident from the definition of the integral as the limit of a sum,

$$\int_a^c y\,dx$$

will be negative. Therefore, if y is negative from $x=a$ to $x=c$, and positive from $x=c$ to $x=b$ $(b>c)$, then

$$\int_a^b y\,dx, \text{ i.e., } \int_a^c y\,dx+\int_c^b y\,dx,$$

will be the difference of the numerical values of the last two integrals, and will thus give us the value of the algebraic sum of the areas above and below the x-axis, when areas above the x-axis are regarded as positive and those below as negative. If, therefore, the numerical sum of the areas is required, these integrals must be evaluated separately. For example, it is easy to see that the curve

$$a^2y^2=x^2(a^2-x^2)$$

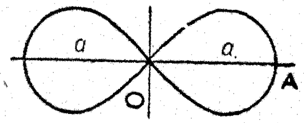
is symmetrical about both the axes and consists of two loops.

One might be tempted to say that the whole area

$$=2\times\text{area above the } x\text{-axis}$$

$$=2\int_{-a}^{a} y\,dx=\frac{2}{a}\int_{-a}^{a} x\sqrt{(a^2-x^2)}\,dx$$

$$=\frac{1}{a}\Big[-\tfrac{2}{3}(a^2-x^2)^{3/2}\Big]_{-a}^{a}=0.$$



But the area above the x-axis is certainly not zero. The reason why we get zero as the value of the integral is that $x\sqrt{(a^2-x^2)}$, and so also $(a^2-x^2)^{3/2}$, has two values; and if we take the positive square root, $x\sqrt{(a^2-x^2)}$ is negative from $x=-a$ to $x=0$. So we have really found above the algebraic sum of the areas in the first and third quadrants.

The easiest plan in such cases is to find the area of the smallest part which by considerations of symmetry will give the area of the whole curve. Thus, in the present example, we can find the area which lies in the first quadrant and multiply it by 4.

(iii) When $F(x)$, the integral of $f(x)$, involves an inverse function, some care is required in finding its values at a and b, because such a function is many-valued. The easiest method is to put down for the value of the function at the lower limit the principal value of the function, i.e., the value between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ (both values inclusive) for $\sin^{-1}x$,

$\tan^{-1}x$, $\cot^{-1}x$, and $\operatorname{cosec}^{-1}x$, and the value between 0 and π (both values inclusive) for $\cos^{-1}x$, $\sec^{-1}x$ and $\operatorname{vers}^{-1}x$. To select the value at the upper limit, remember that

$$\Big[F(x)\Big]_a^x$$

is a continuous function of x, which is zero when $x=a$. Consider, therefore, how $F(x)$ changes from $F(a)$ as x increases from a to b, and choose for $F(b)$ that value at which $F(x)$ will arrive by varying continuously as x varies from a to b.

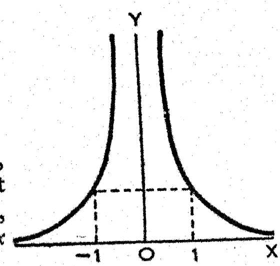
These considerations are applicable also in the evaluation of definite integrals not connected primarily with areas.

For an example illustrating this point see Ex. 20 on page 145.

(iv) If the ordinate becomes infinite at any point between $x=a$ and $x=b$, the method of § 5·6 should be applied; otherwise some absurd result may be arrived at. Thus for the curve

$$y=1/x^2,$$

which lies entirely above the x-axis, one may be tempted to say that the area included between the curve, the x-axis and the ordinates at $x=-1$ and $x=1$



$$=\int_{-1}^{1}\frac{1}{x^2}\,dx=\Big[-\frac{1}{x}\Big]_{-1}^{1}=-1-1=-2.$$

This is certainly wrong, because $1/x^2$ is positive for all values of x; hence

$$\int_{-1}^{1}\frac{1}{x^2}\,dx$$

should be positive. Even numerically the area under consideration cannot be equal to 2, because the area of the rectangular part marked in the figure by a dotted line, is equal to 2; and certainly the area required is greater.

The error lies in ignoring the fact that the ordinate $\to\infty$ as $x\to 0$. The above procedure is just as wrong as, say, summing up $1+2+2^2+2^3+\ldots$ by the following method:

Let $$s=1+2+2^2+2^3+\ldots.$$

Then, multiplying by 2, $$2s=\quad 2+2^2+2^3+\ldots.$$

Therefore, subtracting the first from the second, we have

$$s=-1.$$

We must consider such integrals by the method of limits. If we apply this method to the present case, we find that

$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$

has no meaning. (See § 5·6, Ex. 6.)

(v) It is sometimes convenient, in finding the area of a curve, to utilise the fact that the area included between a curve, the y-axis and the lines $y=a$ and $y=b$ $(b>a)$ is equal to

$$\int_a^b x\,dy.$$

(vi) The process of finding an area is often called *quadrature*.

EXAMPLES

Find the area bounded by the axis of x, and the following curve and ordinates:

1. $y=e^x$; $x=a$, $x=b$.
2. $y=c\cosh(x/c)$; $x=0$, $x=a$. [*Utkal*, 1956]
3. $y=\log x$; $x=a$, $x=b(b>a>1)$.
4. $y=\sin^2 x$; $x=0$, $x=\frac{1}{2}\pi$.
5. Trace the curve $a^2y=x^2(x+a)$ and show that the curve includes with the axis of x an area $a^2/12$. [*Alld.*, '57]
6. Show that the area cut off a parabola by any double ordinate is two-thirds of the corresponding rectangle contained by that double ordinate and its distance from the vertex. [*Agra*, 1940]

7. Show how the area of a semicircle can be expressed as a definite integral.

8. Calculate the area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$ [*Baroda*, 1960]

9. Trace the curve $ay^2=x^2(a-x)$ and show that the area of its loop is $\frac{8}{15}a^2$. [*Aligarh*, 1956]

10. Find the area of the loop of the curve

$$a^3y^2=x^4(b+x).$$ [*Allahabad*, 1957]

11. Find the whole area of the curve

$$a^2y^2=x^3(2a-x).$$ [*Ujjain*, 1960]

12. Trace the curve $a^2y^2=a^2x^2-x^4$ and find the whole area within it. [*Bihar*, 1954]

13. Trace the curve $a^4y^2=x^5(2a-x)$ and prove that its area is to that of the circle whose radius is a, as 5 to 4.

14. Trace the curve $x^2y^2=a^2(y^2-x^2)$, and find the whole area included between the curve and its asymptotes. [*Jammu*, 1954]

15. Find the whole area of the curve

$$a^2x^2=y^3(2a-y).$$ [*Banaras*, 1959]

16. Find the area of the segment cut off from the parabola $y^2=2x$ by the straight line $y=4x-1$. [*Jammu*, '56]

17. Find the area common to the two curves

$$y^2=ax, \quad x^2+y^2=4ax.$$ [*Rajasthan*, 1959]

18. Show that the larger of the two areas into which the circle $x^2+y^2=64a^2$ is divided by the parabola $y^2=12ax$ is

$$\frac{16}{3}a^2(8\pi-\sqrt{3}).$$ [*Allahabad*, 1960]

19. Find the whole area of the curve given by the equations $x=a\cos^3 t$, $y=b\sin^3 t$. [*Panjab*, 1954]

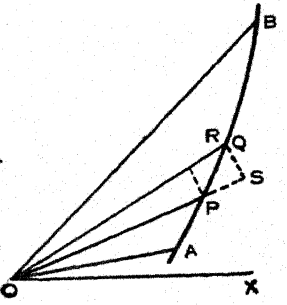
20. Show that the area bounded by the cissoid

$$x=a\sin^2 t,\ y=a(\sin^3 t)/\cos t$$

and its asymptote is $3\pi a^2/4$. [*Allahabad*, 1959]

6·2. Areas of curves given by polar equations. Let $f(\theta)$ be continuous for every value of θ in the domain (α, β). Then the area bounded by the curve $r=f(\theta)$ and the radii vectores $\theta=\alpha$, $\theta=\beta$ $(\alpha<\beta)$, is equal to

$$\frac{1}{2}\int_{\alpha}^{\beta} \mathbf{r}^2\, \mathbf{d}\theta.$$



We can prove this proposition either by dividing the area into a number of sectors and finding the limit of the sum of their areas, or by finding the differential coefficient of the area between the curve, the radius vector $\theta=\alpha$ and a general radius vector. We adopt here the former procedure.

Let AB be the curve, OA and OB the radii vectores $\theta=\alpha$ and $\theta=\beta$.

Divide $\beta-\alpha$ into n parts each equal to h and draw the corresponding radii vectores. (In the figure all the radii vectores are not drawn for the sake of clearness.) Let P and Q be the points on the curve corresponding to $\theta=\alpha+mh$ and $\theta=\alpha+(m+1)h$, and suppose that r goes on increasing as θ increases from α to β. (This restriction is removed below).

With centre O and radii OP, OQ respectively, draw arcs PR, QS as in the figure. Then the area OPQ lies in magnitude between

$$\tfrac{1}{2}OP^2 \,.\, h \quad \text{and} \quad \tfrac{1}{2}OQ^2 \,.\, h,$$

i.e., between $\frac{1}{2}[f\{\alpha+mh\}]^2 h$ and $\frac{1}{2}[f\{\alpha+(m+1)h\}]^2 h$.

Hence the area AOB lies between

$$\tfrac{1}{2}\sum_{m=0}^{n-1}[f\{\alpha+mh\}]^2 h \quad \text{and} \quad \tfrac{1}{2}\sum_{m=0}^{n-1}[f\{\alpha+(m+1)h\}]^2 h.$$

Take limits as $n \to \infty$. Then, as the limit of each of the two sums last written is

$$\frac{1}{2}\int_{\alpha}^{\beta} \{f(\theta)\}^2 \, d\theta,$$

it follows that the area AOB is also equal to this definite integral.

If r decreases all the way from $\theta=\alpha$ to $\theta=\beta$, then also the above proof is evidently applicable. But if there are a finite number of points of maxima and minima between α and β, say at $\theta=\theta_1, \theta_2 \ldots, \theta_p$, then by drawing the corresponding radius vectores we can divide the area AOB into $p+1$ sectors, to each of which the above formula is certainly applicable. Hence the total area AOB is

$$\frac{1}{2}\int_{\alpha}^{\theta_1} r^2 \, d\theta + \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta + \ldots + \frac{1}{2}\int_{\theta_{p-1}}^{\theta_p} r^2 \, d\theta + \frac{1}{2}\int_{\theta_p}^{\beta} r^2 \, d\theta$$

i.e.,

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta.$$

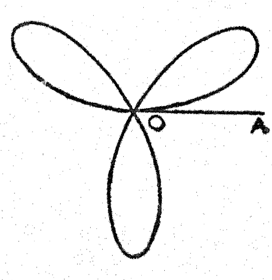
NOTE 1. If the radius vector tends to ∞ as $\theta \to \gamma$, where $\alpha < \gamma < \beta$, then the method of § 5·6 should be applied.

2. In some cases it is more convenient to transform a given Cartesian equation into polars than to solve for y. In such cases the formula of the present article should be applied after transformation to polars.

Ex. 1. Find the area of a loop of the curve $r=a \sin 3\theta$.

[*Allahabad*, 1960]

One loop is obtained by the values of θ from $\theta=0$ to $\theta=\frac{1}{3}\pi$.



Hence the area of a loop

$$=\frac{1}{2}a^2 \int_0^{\pi/3} \sin^2 3\theta \, d\theta$$

$$=\frac{1}{6}a^2 \int_0^{\pi} \sin^2 \phi \, d\phi,$$

where $3\theta=\phi$,

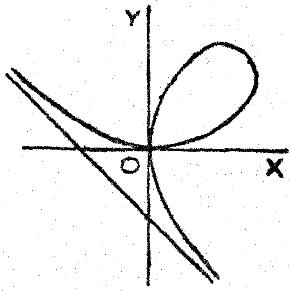
$$=\frac{2}{6}a^2 \int_0^{\pi/2} \sin^2 \phi \, d\phi, \text{ by § 5·2,}$$

$$=\tfrac{2}{3}a^2\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{2\Gamma(2)}=\tfrac{1}{6}a^2\cdot\tfrac{1}{2}\cdot\pi=\tfrac{1}{12}\pi a^2.$$

Ex. 2. Find the area of the loop of the folium $x^3+y^3-3axy=0.$ [*Lucknow*, 1962]

Changing to polars,

$$r=\frac{3a\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}.$$



The loop extends from $\theta=0$ to $\theta=\frac{1}{2}\pi$. Hence the required area

$$=\frac{9a^2}{2}\int_0^{\pi/2}\frac{\sin^2\theta\cos^2\theta\,d\theta}{(\cos^3\theta+\sin^3\theta)^2}$$

$$=\frac{9a^2}{2}\int_0^{\pi/2}\frac{\tan^2\theta\sec^2\theta\,d\theta}{(1+\tan^3\theta)^2}$$

$$=\frac{3a^2}{2}\int_0^{\infty}\frac{dt}{(1+t)^2},\text{ where } t=\tan^3\theta,=\frac{3a^2}{2}\left[\frac{-1}{1+t}\right]_0^{\infty}=\tfrac{3}{2}a^2.$$

Examples

Find the area between the following curves and radii vectores :

1. The spiral $r\theta^{1/2}=a$; $\theta=\alpha$, $\theta=\beta$.

2. The parabola $l/r=1+\cos\theta$; $\theta=0$, $\theta=\alpha$.

3. The equiangular spiral $r=ae^{m\theta}$; $\theta=\alpha$, $\theta=\beta$.

4. Find the area of one loop of $r=a\cos 4\theta$.

5. Calculate the ratio of the area of the larger to the area of the smaller loop of the curve

$$r=\tfrac{1}{2}+\cos 2\theta.$$ [*Nagpur*, 1956]

6. Find the area of the curve $r^2=a^2\cos^2\theta+b^2\sin^2\theta$. [*Gauhati*, 1949]

7. Find the area of the loop of the curve $r=a\theta\cos\theta$ between $\theta=0$ and $\theta=\frac{1}{2}\pi$. [*Panjab*, 1951]

8. Find the area bounded by the curve $r=a(1+\cos\theta)$. [*Lucknow*, 1957]

9. Trace the curve $r=3+2\cos\theta$ and find the area enclosed.

10. Trace the curve $r=\sqrt{3}\cos 3\theta+\sin 3\theta$, and find the area of a loop. [*Lucknow*, 1948]

11. Prove that the sum of the areas of the two loops of the limaçon, $r=a+b\cos\theta$, $b>a$, is equal to

$\frac{1}{2}\pi(2a^2+b^2)$. [*Aligarh*, 1957]

12. Show that the area of a loop of $r=a\cos n\theta$ is $\pi a^2/4n$, n being integral. Also prove that the whole area is $\frac{1}{4}\pi a^2$ or $\frac{1}{2}\pi a^2$ according as n is odd or even.

13. Show that the area of a loop of the curve

$$x(x^2+y^2)=a(x^2-y^2)$$

is $2a^2(1-\frac{1}{4}\pi)$. [*Baroda*, 1960]

14. Find the area of a loop of the curve

$$x^4+y^4=4a^2xy.$$ [*Agra*, 1953]

15. Prove that the area between the cissoid

$$r=a\sin^2\theta/\cos\theta$$

and its asymptote is $\frac{3}{4}\pi a^2$. [*Banaras, Eng.*, 1958]

16. Find the area common to the circles

$r=a\sqrt{2}$ and $r=2a\cos\theta$. [*Ujjain*, 1960]

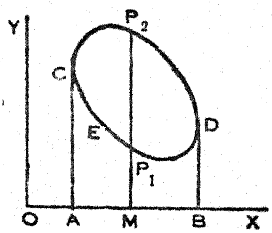
6·3. Area of closed curves.

We have found the areas of many closed curves by an application of § 6·1 or § 6·2. When, however, a closed curve is given by equations of the form

$$x=f_1(t), \quad y=f_2(t),$$

where $f_1(t)$ and $f_2(t)$ are single-valued functions of t, another procedure is generally simpler.

Let CP_1DP_2 be a closed curve given by $x=f_1(t), y=f_2(t)$, and let it be cut by a line parallel to the y-axis in two (and only two) points P_1, P_2. Let $MP_2>MP_1$.



Let AC and BD be the tangents parallel to the y-axis, and let $OA=a$, $OB=b(b>a)$.

Then the area of the figure CP_1DP_2=the area of the figure $ABDP_2C$

—the area of the figure $ABDP_1C$

$$=\int_a^b MP_2\,dx-\int_a^b MP_1\,dx \quad . \quad . \quad (1)$$

$$=-\int_b^a MP_2\,dx-\int_a^b MP_1\,dx.$$

Suppose that as t increases from t_1 to t_2, the point (x, y) travels from a point E on the curve back to the point E, *via* P_1, D, P_2, C taken in order. Let the values of t corresponding to D, C be t_d, t_c respectively.

Then $$-\int_a^b MP_1\,dx=-\int_{t_1}^{t_d} y\frac{dx}{dt}\,dt-\int_{t_c}^{t_2} y\frac{dx}{dt}\,dt,$$

and $$-\int_b^a MP_2\,dx=-\int_{t_d}^{t_c} y\frac{dx}{dt}\,dt.$$

Hence, by addition, the area CP_1DP_2

$$=-\int_{t_1}^{t_2} y\frac{dx}{dt}\,dt, \text{ by § 5·2.}$$

We can show similarly, by drawing the tangents parallel to the x-axis, and applying the formula $\int x\,dy$ for the area bounded by a curve, the y-axis

and two straight lines parallel to the x-axis, that the area CP_1DP_2

$$=\int_{t_1}^{t_2} x\frac{dy}{dt}\,dt.$$

Adding the two expressions thus obtained for the area and dividing by 2, we get

$$\textbf{area } \mathbf{CP_1DP_2}=\tfrac{1}{2}\int_{t_1}^{t_2}\left(\mathbf{x}\frac{\mathbf{dy}}{\mathbf{dt}}-\mathbf{y}\frac{\mathbf{dx}}{\mathbf{dt}}\right)\mathbf{dt}.$$

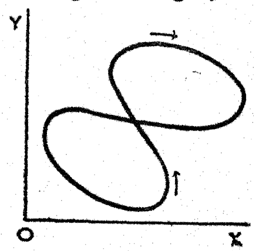
This is often abbreviated into the formula:

$$\textbf{area } \mathbf{CP_1DP_2}=\tfrac{1}{2}\int(\mathbf{xdy}-\mathbf{ydx}),\ \textit{taken round the curve.}$$

In the above we have supposed that the curve is described in the direction CP_1DP_2 as t increases, and takes for the area the value (1) which is positive, i.e., we have considered the *area to be positive when it lies to the left of an observer moving along the curve in the direction corresponding to increasing t.* This is the usual convention.

The formula will give a negative value for an area described in the opposite direction.

If a curve crosses itself so as to form a figure of eight, the two loops of which are described, as t increases, in such a way that one loop lies to the left, and the other to the right, of the observer moving along the curve in the direction of increasing t, the above formula will give us the difference of the areas of the two loops.



The formula is true even when the curve cuts one or both the axes.

Ex. Find the area of the ellipse $x=a\cos t$, $y=b\sin t$.

Here $x\dfrac{dy}{dt}=a\cos t\,.\,b\cos t$, $y\dfrac{dx}{dt}=-a\sin t\,.\,b\sin t$.

Hence the area of the ellipse $=\frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$

$$=\tfrac{1}{2}ab\int_0^{2\pi}(\cos^2 t+\sin^2 t)\,dt=\pi ab.$$

[Notice that as t increases, the corresponding point on the curve moves in such a way that the area lies to the left of an observer moving with the point; also that the formula has given us a positive value for the area. This confirms the above remarks regarding signs.]

EXAMPLES

1. Show that the area of the loop of the curve

$$x=a(1-t^2),\ y=at(1-t^2),\ -1\leqslant t\leqslant 1,$$

is $\frac{8}{15}a^2$. [*Allahabad*, 1947]

2. A closed curve is defined by the equations

$$x=\phi(t),\quad y=\psi(t);$$

prove that its area is given by $\frac{1}{2}\int\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$, certain conventions being adopted.

Find the area of the loop of the curve

$$x=\frac{a\sin 3\theta}{\sin\theta},\ y=\frac{a\sin 3\theta}{\cos\theta}.$$ [*Allahabad*, '57]

3. Prove that the area of the curve

$$x=a\cos\theta+b\sin\theta+c,\quad y=a'\cos\theta+b'\sin\theta+c'$$

is equal to $\pi(ab'-a'b)$. [*Karnatak*, 1954]

6·4. Simpson's rule. There are many formulae* which enable us to evaluate approximately an area, or a definite integral, even when the analytical relation between y and x is not known. All that is required is that the values of y should

*See Whittaker and Robinson: *The Calculus of Observations* (Blackie and Son).

be known for values of x sufficiently close to one another.

One such formula is Simpson's rule* which states that *an approximate value of*

$$\int_a^b y\,dx$$

is
$$\tfrac{1}{3}h\{y_1+y_{2n+1}+2(y_3+y_5+\dots+y_{2n-1})+4(y_2+y_4+\dots+y_{2n})\},$$

where n is any positive number, $h=(b-a)/2n$ and y_r is the value of y corresponding to the value $a+(r-1)h$ of x.

To prove this, assume that the functional relation between y and x is $y=f(x)$. Let $P_1, P_2, P_3, \dots$ be the points on the curve $y=f(x)$ whose abscissae are a, $a+h$, $a+2h$, etc. Let us suppose that the curve (parabola) whose equation referred to $(a+h, 0)$ as origin is of the form

$$y=A+Bx+Cx^2,$$

and which passes through P_1, P_2 and P_3, is a sufficiently close approximation† to the curve $y=f(x)$ between P_1 and P_3.

To determine A, B, C, we have the equations:

$$y_1=A-Bh+Ch^2,$$

$$y_2=A,$$

$$y_3=A+Bh+Ch^2,$$

*Named after Thomas Simpson (1710-1761), an able and self-taught English mathematician, for many years professor at the Royal Military Academy at Woolwich and author of several text-books(Cajori : *A History of Mathematics*). The formula, however, had been discovered much earlier in a geometrical form by B. Cavalieri (1598-1647).

†Evidently, the smaller h is, the closer will be the approximation. It is assumed, of course, that $y=f(x)$ is some smooth curve passing through $P_1, P_2, \dots, P_{2n+1}$.

which express the condition that P_1, P_2, P_3, are on the assumed parabola.

Now the area between the curve $P_1P_1P_3$, the axis of x and the ordinates of P_1, P_3 is

$$\int_{-h}^{h}(A+Bx+Cx^2)\,dx, \quad \text{i.e.,} \quad 2h\{A+\tfrac{1}{3}Ch^2\}.$$

Substituting in this the values of A and C obtained from the above equations, we find that the area under consideration is

$$\tfrac{1}{3}h(y_1+4y_2+y_3).$$

Similarly the area between the parabolic arc $P_3P_4P_5$, the axis of x and the ordinates of P_3, P_5 is

$$\tfrac{1}{3}h(y_3+4y_4+y_5),$$

and so on.

By addition, we find that an approximate value of the area required is

$$\tfrac{1}{3}h\{(y_1+4y_2+y_3)+(y_3+4y_4+y_5)+\dots+(y_{2n-1}+4y_{2n}+y_{2n+1})\},$$

i.e., $\quad \tfrac{1}{3}h\{y_1+y_{2n+1}+2(y_3+y_5+\dots+y_{2n-1})+4(y_2+y_4+\dots+y_{2n})\}.$

Ex. 1. Given that $e^0=1$, $e^1=2{\cdot}72$, $e^2=7{\cdot}39$, $e^3=20{\cdot}09$, $e^4=54{\cdot}60$, verify Simpson's rule by finding an approximate value of

$$\int_0^4 e^x\,dx,$$

and comparing it with the exact value. [*Panjab*, 1954]

By Simpson's formula, an approximate value is

$$\tfrac{1}{3}\{1+54{\cdot}60+2(7{\cdot}39)+4(2{\cdot}72+20{\cdot}09)\}=53{\cdot}87.$$

The correct value $=\Big[e^x\Big]_0^4=54{\cdot}60-1=53{\cdot}60.$

We see that the approximation is pretty close, although h is by no means small.

EXAMPLES

1. Taking 10 equal parts, show by Simpson's rule, that

$$\int_0^1 \frac{\log(1+x^2)}{1+x^2}\,dx = 0 \cdot 1728. \qquad \text{[Lucknow, 1957]}$$

2. If $y=f(x)=a+bx+cx^2$, $y_1=f(p)$, $y_2=f(p+h)$ and $y_3=f(p+2h)$, prove that

$$\int_p^{p+2h} f(x)\,dx = \tfrac{1}{3}h(y_1+y_3+4y_2).$$

Hence obtain an approximate value of

$$\int_0^{0\cdot 2} (1-2x^2)^{1/3}\,dx. \qquad \text{[Allahabad, 1957]}$$

3. A curve is drawn to pass through the points given by the following table :—

x	1	1·5	2	2·5	3	3·5	4
y	2	2·4	2·7	2·8	3	2·6	2·1

Estimate the area bounded by the curve, the x-axis and the lines $x=1$, $x=4$. [*Baroda*, 1960]

4. Explain Simpson's rule for approximate integration. Use the rule, taking five ordinates, to find an approximation to two decimal places to the value of the integral

$$\int_1^2 \sqrt{(x-1/x)}\,dx. \qquad \text{[Panjab, 1950]}$$

5. A river is 80 feet wide. The depth d in feet at a distance x feet from one bank is given by the following table:

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Find approximately the area of the cross-section.
[*Rajasthan*, 1962]

6. By taking on a curve points P_1, P_2, P_3, ..., whose abscissae are a, $a+h$, $a+2h$, ... $(b-a=nh)$, and joining

P_1P_2, P_3P_4, ... by straight lines, show that an approximate value of

$$\int_a^b f(x)\,dx$$

is $\frac{1}{2}h\{y_1+y_{n+1}+2(y_2+y_3+\ldots+y_n)\}$.

[This is known as the *Trapezoidal Rule*].

7. Prove that the area between the axis of x and the curve

$$y=a+bx+cx^2+dx^3$$

from $x=0$ to $x=3h$ is

$$\tfrac{3}{8}h(y_0+3y_1+3y_2+y_3),$$

where y_r is the value of y when $x=rh$.

Hence find an approximation to the value of the integral

$$\int_0^{\pi/6}(1+6\sin\theta)^{1/2}\,d\theta.$$

Examples on Chapter VI

1. If a, b are positive and $a>b$, prove that the area between the hyperbola $xy=c^2$, the x-axis and the ordinates at a and b is $c^2\log(a/b)$.

If, instead of a hyperbola, the curve is that given by $y=x^n/c^{n-1}$, prove that the area is

$$\frac{a^{n+1}-b^{n+1}}{(n+1)c^{n-1}}.$$

2. Trace the curve $x^{2/3}+y^{2/3}=a^{2/3}$ and find the area enclosed by it. [*Patna*, 1958]

3. Trace the curve

$$y^2=x^3/(2a-x)$$

and show that the entire area between the curve and its asymptote is $3\pi a^2$. [*Bihar*, 1956]

4. Trace the curve $y^2(a-x)=x^2(a+x)$ and find the area of the loop. [*Ban. Geophs.*, 1961]

5. Show that the area included between the parabolas

$$y^2=4a(x+a),\ y^2=4b(b-x)$$

is $$\tfrac{8}{3}(a+b)\sqrt{(ab)}.$$ [*Karnatak*, 1959]

6. Show that the area common to the ellipses

$$a^2x^2+b^2y^2=1,\ b^2x^2+a^2y^2=1, \text{ where } 0<a<b,$$

is $4(ab)^{-1}\tan^{-1}(a/b)$. [*Nagpur*, 1961]

7. Find the area included between the curves $y^2=4ax$ and $x^2=4ay$. [*Aligarh*, 1960]

8. If A is the vertex, O the centre, and P any point on the hyperbola $x^2/a^2-y^2/b^2=1$, prove that

$$x=a\cosh(2S/ab),$$

and

$$y=b\sinh(2S/ab),$$

where S is the sectorial area OPA. [*Gorakhpur*, 1959]

9. In the case of the cycloid

$$x=a(\theta+\sin\theta),\quad y=a(1-\cos\theta),$$

find the area included between the curve and its base.

10. Prove that the whole area between the four infinite branches of the tractrix

$$x=a\cos t+\tfrac{1}{2}a\log\tan^2\tfrac{1}{2}t,$$
$$y=a\sin t$$

is πa^2. [*Allahabad*, 1957]

11. Prove that the area of the loop of the curve

$$x^3+y^3=3axy$$

is three times the area of one of the loops of the curve

$$r^2=a^2\cos 2\theta.$$ [*Panjab*, 1950]

12. Find the area of a loop of the curve

$$r=a\cos 3\theta+b\sin 3\theta.$$ [*P.S.C., U.P.*, 1959]

13. Prove that the area of the loop of the curve

$x^5+y^5=5ax^2y^2$ is $\frac{5}{2}a^2$. [*Delhi*, 1959]

14. Prove that the area of the loop of the curve

$x^6+y^6=a^2x^2y^2$ is $\pi a^2/12$. [*Aligarh*, 1958]

15. Prove that the area of the curve

$x^4-3ax^3+a^2(2x^2+y^2)=0$ is $\frac{3}{8}\pi a^2$.

16. Trace the curve

$$r=a(\sec\theta+\cos\theta),$$

and find the area between the curve and its asymptote.
[*Rajasthan*, 1957]

17. Find the area lying between the cardioid $r=a(1-\cos\theta)$ and its double tangent.

18. Find the ratio of the two parts into which the parabola $2a=r(1+\cos\theta)$ divides the area of the cardioid $r=2a(1+\cos\theta)$.
[*Raj.*, 1954]

19. Prove that the area of the curve

$$r^2(2c^2\cos^2\theta-2ac\sin\theta\cos\theta+a^2\sin^2\theta)=a^2c^2$$

is equal to πac.

20. Show that the sectorial area of the ellipse $x^2/a^2+y^2/b^2=1$, included between the semi-diameters $\theta=0$ and $\theta=\alpha$, is

$$\tfrac{1}{2}ab\tan^{-1}\left(\frac{a}{b}\tan\alpha\right),\ \text{if } 0\leqslant\alpha\leqslant\pi,$$

$$\tfrac{1}{2}ab\left\{\pi+\tan^{-1}\left(\frac{a}{b}\tan\alpha\right)\right\},\ \text{if } \tfrac{1}{2}\pi\leqslant\alpha\leqslant\pi,$$

where in each case that value of $\tan^{-1}\{(a/b)\tan\alpha\}$ is implied which lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. Hence prove that the area included between a pair of conjugate semi-diameters is $\frac{1}{4}\pi ab$.

21. O is the pole of the lemniscate $r^2=a^2\cos 2\theta$ and PQ is a common tangent to its two loops. Find the area bounded by the line PQ and the arcs OP and OQ of the curve.
[*P.S.C., U.P., Forest*, 1955]

22. Prove that the area included between the folium

$$x^3+y^3=3axy$$

and its asymptote is equal to the area of its loop.
[*P.S.C., U.P.*, 1955]

23. P is any point of the circle $r=2c\sin\theta$, PT the tangent at P, OT the perpendicular from the origin on PT. Determine the area swept out by OT when P describes the circumference of the circle.

24. Prove that the area of a sector of the ellipse of semi-axes a and b between the major axis and a radius vector from the focus is

$$\tfrac{1}{2}ab(\theta-e\sin\theta),$$

where θ is the eccentric angle of the point to which the radius vector is drawn. [*Agra*, 1961]

[Hint. Area of the sector=area of a triangle+area of a segment. Find only the area of the segment by integration.]

25. If the pedal equation of a curve be $p=f(r)$, prove that the area bounded by the curve and two radii vectores is

$$\tfrac{1}{2}\int\frac{pr\,dr}{\sqrt{(r^2-p^2)}},$$

taken between suitable limits. [*Lucknow*, 1960]

CHAPTER VII

LENGTHS OF CURVES

7·1. Lengths of curves. If s denotes the length of the arc of the curve $y=f(x)$, measured from a fixed point A on it, up to any point (x, y) on it, then we know by Differential Calculus that

$$\frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}.$$

It follows at once, as in § 6·1, that

$$s=\int_a^x\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx,$$

where a is the abscissa of the point A from which s is measured.

Hence, if the abscissae of A and B are a and b, the length of the arc AB is equal to

$$\int_a^b \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx.$$

Similarly, if x and y are expressed in terms of a parameter t, and to the points A and B correspond the values t_1 and t_2 of t, then evidently the length of the arc AB is equal to

$$\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}\,dt.$$

Again, since for a curve given by its polar equation,

$$\frac{ds}{d\theta}=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}},$$

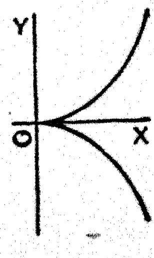
and
$$\frac{ds}{dr}=\frac{ds}{d\theta}\cdot\frac{d\theta}{dr}=\sqrt{\left\{r^2\left(\frac{d\theta}{dr}\right)^2+1\right\}},$$

the length of the arc AB is equal to

$$\int_\alpha^\beta \sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta \quad \text{or} \quad \int_{r_1}^{r_2} \sqrt{\left\{r^2\left(\frac{d\theta}{dr}\right)^2+1\right\}}\,dr,$$

where α and β are the vectorial angles of A and B, and r_1 and r_2 their radii vectores.

Ex. 1. Find the length of the arc of the semicubical parabola $ay^2=x^3$ from the vertex to the point (a, a).



By differentiation

$$2\,ayy'=3x^2.$$

Therefore

$$y'^2=9x^4/4a^2y^2=9x^4/4ax^3=9x/4a.$$

Hence the required length

$$=\int_0^a \sqrt{\{1+9x/4a\}}\,dx$$

$$=\frac{1}{2\sqrt{a}}\int_0^a \sqrt{(4a+9x)}\,dx$$

$$=\frac{1}{2\sqrt{a}}\int_{2\sqrt{a}}^{\sqrt{(13a)}} \frac{t\,.\,2t\,dt}{9}, \text{ where } t^2=4a+9x,$$

$$=\frac{1}{27\sqrt{a}}\left[t^3\right]_{2\sqrt{a}}^{\sqrt{(13a)}}=\tfrac{1}{27}\{13\sqrt{(13)}-8\}a.$$

Ex. 2. Show that $8a$ is the length of an arch of the cycloid whose equations are

$$x=a(t-\sin t), \quad y=a(1-\cos t). \qquad [\textit{Allahabad}, 1960]$$

Here $dx/dt=a(1-\cos t)=2a\sin^2 \frac{1}{2}t,$

$dy/dt=a\sin t=2a\sin\frac{1}{2}t\cos\frac{1}{2}t.$

Also, two consecutive values of t for which $y=0$ are $t=0$ and $t=2\pi$.

Hence the length of an arch

$$=\int_0^{2\pi}\sqrt{\{(2a)^2\sin^4\tfrac{1}{2}t+(2a)^2\sin^2\tfrac{1}{2}t\cos^2\tfrac{1}{2}t\}}\,dt$$

$$=2a\int_0^{2\pi}\sin\tfrac{1}{2}t\,dt=4a\left[-\cos\tfrac{1}{2}t\right]_0^{2\pi}=8a.$$

Ex. 3. Find the length of the arc of the equiangular spiral $r=ae^{\theta\cot\alpha}$ between the points for which the radii vectores are r_1 and r_2. [*Baroda*, 1960]

Here $dr/d\theta=a\cot\alpha\, e^{\theta\cot\alpha}=r\cot\alpha.$

Hence the required length

$$=\int_{r_1}^{r_2}\sqrt{\left\{r^2\left(\frac{d\theta}{dr}\right)^2+1\right\}}\,dr$$

$$=\int_{r_1}^{r_2}\sqrt{\{\tan^2\alpha+1\}}\,dr=\sec\alpha\Big[\,r\,\Big]_{r_1}^{r_2}=(r_2-r_1)\sec\alpha.$$

7·11. Remarks. (i) It must be remembered that

$$\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$$

has two values, $\pm\phi(x)$, where $\{\phi(x)\}^2=1+(dy/dx)^2$. For finding lengths, the positive value must be chosen. If

the negative value is chosen, the length will obviously come out negative.

If $\phi(x)$ is a function which changes sign at some value c within the range of integration (a, b), then the definite integral from a to b must be broken into the sum of two definite integrals, one from a to c and the other from c to b, and the positive value of the integrand taken in each. Otherwise the result will be the difference of the lengths of the two arcs.

Thus, if the length of the arc of the cycloid

$$x=a(t-\sin t), y=a(1-\cos t),$$

is wanted from $t=0$ to $t=4\pi$ (two complete arches), one might be tempted to work thus :

$$\text{Required length} =\int_0^{4\pi} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}\, dt$$

$$=2a\int_0^{4\pi} \sin \tfrac{1}{2}t\, dt, \text{ as in Ex. 2, § 7·1,}$$

$$=4a\Big[-\cos \tfrac{1}{2}t\Big]_0^{4\pi}=0.$$

The reason why we get zero is that $\sin \frac{1}{2}t$ is negative from $t=2\pi$ to $t=4\pi$.

The easiest procedure is to find the length of the smallest part which by symmetry will give the whole length wanted, and see that the integrand is not negative within the range of integration.

(ii) Finding the length of a curve is also called *rectification.*

(iii) If it is more convenient in any particular case to take y as the independent variable, we can use the formula

$$s=\int \sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\, dy.$$

EXAMPLES

Find the length of

1. The arc of the parabola $x^2=4ay$ from the vertex to an extremity of the latus-rectum. [*Allahabad*, 1959]

2. The arc of the curve $y=ae^x$ from the point $(0, a)$ to the point (x_1, y_1).

3. Show that the length of the curve $y=\log \sec x$ between the points where $x=0$ and $x=\frac{1}{3}\pi$ is $\log_e(2+\sqrt{3})$.

4. Find the length of an arc of the parabola $y=x^2$, measured from the vertex.

Calculate the length of the arc to the point $(1, 1)$, given $\log_e(2+\sqrt{5})=1{\cdot}45$.

5. Prove that the length and area of the loop of the curve $3ay^2=x(x-a)^2$ are $4a/\sqrt{3}$ and $8a^2/15\sqrt{3}$ respectively. [*First part, Osmania*, 1957]

6. Find the length of the curve

$$y=\log\frac{e^x-1}{e^x+1}$$

from $x=1$ to $x=2$. [*B. H. U.* 1962]

7. Show that the length of an arc of the curve

$$x^2=a^2(1-e^{y/a})$$

measured from $(0, 0)$ to (x, y) is $a\log\dfrac{a+x}{a-x}-x$. [*Lucknow*, 1951]

8. Show that in the epicycloid for which

$$x=(a+b)\cos\theta-b\cos\{(a+b)\,\theta/b\},$$
$$y=(a+b)\sin\theta-b\sin\{(a+b)\,\theta/b\},$$
$$s=\{4b(a+b)/a\}\cos(a\theta/2b),$$

s being measured from the point at which $\theta=\pi b/a$.

9. Rectify the curve

$$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta).$$ [*Delhi*, '45]

10. If the coordinates of a point on the curve $3ay^2=x(a-x)^2$ are expressed in terms of a parameter in the form

$$x=3at^2,\quad y=a(t-3t^3),$$

show how the curve is traced out as t increases from $-\infty$ to $+\infty$.

Show that the length of the arc of this curve from the origin to the point of the loop where the tangent to the curve makes an angle ψ with the y-axis, is

$$\tfrac{1}{3}a(3\tan\tfrac{1}{2}\psi+\tan^3\tfrac{1}{2}\psi).$$

Find the length of

11. The arc of the cardioid $r=a(1-\cos\theta)$ between the points whose vectorial angles are α and β. [*Allahabad*, 1957]

12. The perimeter of the cardioid $r=a(1-\cos\theta)$. [*Delhi*, 1959]

13. The arc of the spiral $r=a\theta$ between the points whose radii vectores are r_1 and r_2. [*Nagpur*, 1961]

14. The perimeter of the curve $r=a\cos\theta$. [*Alig.*, '45]

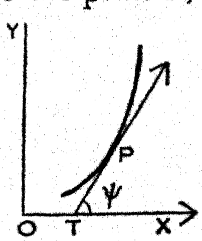
15. Find the length of the curve $r^{1/3}=8\cos\frac{1}{3}\theta$.

16. Find the entire length of the cardioid $r=a(1+\cos\theta)$, and show that the arc of the upper half is bisected by $\theta=\frac{1}{3}\pi$. [*Banaras*, 1959]

17. Find the length of an arc of the cissoid

$$r=a\sin^2\theta/\cos\theta.$$ [*Jammu*, 1953]

7·2. Intrinsic equations. By the *intrinsic equation* of a curve is meant a relation between s and ψ where s is the length of the arc AP of a curve measured from a fixed point A up to the point P, and ψ is the angle which the tangent to the curve at P makes with a fixed straight line (generally the tangent at A).



(i) *To find the intrinsic equation from the Cartesian equation.* Take the axis of x as the fixed straight line with reference to which ψ is measured. Let the abscissa of the

point from which s is measured be a. Then, if the equation of the curve is $y=f(x)$, we have

$$\tan\psi=df(x)/dx, \quad . \quad . \quad (1)$$

and
$$s=\int_a^x \sqrt{\{1+y'^2\}}\,dx$$
$$=F(x), \text{ say.} \qquad (2)$$

Eliminating x between (1) and (2), we get a relation between s and ψ, which is the intrinsic equation of the curve.

Ex. Find the intrinsic equation of the catenary
$$y=a\cosh(x/a). \qquad [\textit{Gorakhpur}, 1959]$$

Suppose s is measured from the point whose abscissa is 0.

Then
$$s=\int_0^x \sqrt{\{1+\sinh^2(x/a)\}}\,dx$$
$$=\int_0^x \cosh(x/a)\,dx=a\sinh(x/a).$$

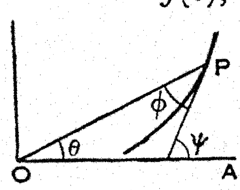
Also $\tan\psi=dy/dx=\sinh(x/a)$.

Hence the required intrinsic equation is
$$s=a\tan\psi.$$

(ii) *To find the intrinsic equation from the Polar equation.* Take the initial line as the fixed straight line with reference to which ψ is measured. Let α be the vectorial angle of the fixed point from which s is measured. Then, if the curve be $r=f(\theta)$,

$$\psi=\theta+\phi \quad . \quad . \quad . \quad (1)$$

$$\tan\phi=r\frac{d\theta}{dr}=\frac{f(\theta)}{f'(\theta)}, \quad . \quad . \quad (2)$$



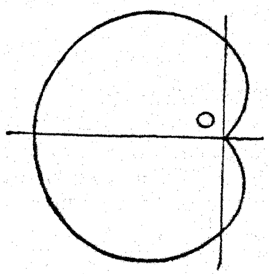
$$\text{and } s=\int_\alpha^\theta \sqrt{[\{f(\theta)\}^2+\{f'(\theta)\}^2]}\,d\theta$$
$$=F(\theta), \text{ say.} \quad . \quad . \quad . \quad (3)$$

Eliminating θ and ϕ between (1), (2) and (3), we get a relation between s and ψ, which is the intrinsic equation of the curve.

Ex. Find the intrinsic equation of the cardioid

$$r=a(1-\cos\theta).\qquad [Allahabad, 1958]$$

Suppose s is measured from the pole.



Then

$$s=a\int_0^\theta \sqrt{\{(1-\cos\theta)^2+\sin^2\theta\}}\,d\theta$$

$$=2a\int_0^\theta \sin\tfrac{1}{2}\theta\,d\theta=4a\Big[-\cos\tfrac{1}{2}\theta\Big]_0^\theta$$

$$=4a(1-\cos\tfrac{1}{2}\theta)=8a\sin^2\tfrac{1}{4}\theta.$$

Also $\tan\phi=r\,d\theta/dr$

$$=(1-\cos\theta)/\sin\theta=\tan\tfrac{1}{2}\theta.$$

Therefore $\phi=\frac{1}{2}\theta$, so that $\psi=\frac{3}{2}\theta$.

It follows that $s=8a\sin^2\frac{1}{6}\psi$, which is the required intrinsic equation.

7·3. Length of arc of an evolute. If a curve is given and the length of an arc of its evolute is required, it is not necessary to find the evolute first. We can use the following proposition (see *Text-Book on Diff. Cal.*, § 12·42).

The difference between the radii of curvature at any two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.

Examples

1. Show that the intrinsic equation of the parabola $y^2=4ax$ is

$$s=a\cot\psi\operatorname{cosec}\psi+a\log(\cot\psi+\operatorname{cosec}\psi),$$

ψ being the angle between the x-axis and the tangent at the point whose distance from the vertex is s.

2. Show that the intrinsic equation of the equiangular spiral $r=ae^{m\theta}$, when the arc is measured from $(a, 0)$, is

$$s=a\sqrt{(1+m^2)}\,.\{e^{m(\psi-\beta)}-1\}/m,$$

where $\beta=\tan^{-1}(1/m)$.

3. Show that in the parabola $2a/r=1+\cos\theta$,

$$\frac{ds}{d\psi}=\frac{2a}{\sin^3\psi}.$$ [*Gorakhpur*, 1960]

4. Show that the whole length of the evolute of the ellipse $x^2/a^2+y^2/b^2=1$ is $4(a^2/b-b^2/a)$.

5. Find the evolute of a parabola, and show that the length of the arc of the evolute from the cusp to the point at which the evolute meets the parabola is $2a(3\sqrt{3}-1)$, where $4a$ is the latus rectum of the parabola. [*Lucknow*, 1951]

6. Find the intrinsic equation of the cardioid

$$r=a(1+\cos\theta),$$

and hence, or otherwise, prove that

$$s^2+9\rho^2=16a^2,$$

where ρ is the radius of curvature at any point, and s is the length of the arc intercepted between the vertex and the point. [*Banaras*, 1958]

Examples on Chapter VII

1. Show that the length of the arc of the parabola $y^2=4ax$ cut off by the line $3y=8x$ is $a(\log_e 2+\frac{15}{16})$.

2. If s be the length of the arc of the catenary $y=c\cosh(x/c)$ from the vertex $(0, c)$ to the point (x, y), show that $s^2=y^2-c^2$. [*Allahabad*, 1957]

3. In the catenary $y=a\cosh(x/a)$, prove that the area between the curve, the axis of x and the ordinates of two points on the curve varies as the length of the intervening arc. [*Lucknow*, 1958]

4. In the ellipse $x=a\cos\phi, y=b\sin\phi$, show that

$$ds=a\sqrt{(1-e^2\cos^2\phi)}\,d\phi,$$

and hence show that the whole length of the ellipse is

$$2a\pi\left[1-\left(\frac{1}{2}\right)^2\cdot\frac{e^2}{1}-\left(\frac{1.3}{2.4}\right)^2\cdot\frac{e^4}{3}-\left(\frac{1.3.5}{2.4.6}\right)^2\cdot\frac{e^6}{5}-\text{etc.}\right],$$

where e is the eccentricity of the ellipse. [*Agra*, 1956]

5. Find the length of the arc of the curve

$$x=e^{\theta}\sin\theta,\quad y=e^{\theta}\cos\theta$$

from $\theta=0$ to $\theta=\frac{1}{2}\pi$. [*Allahabad*, 1959]

6. Show that the length of an arc of the curve

$$x\sin\theta+y\cos\theta=f'(\theta),$$
$$x\cos\theta-y\sin\theta=f''(\theta),$$

is given by $\qquad s=f(\theta)+f''(\theta)+C.$ [*Gorakhpur*, 1959]

7. In the four-cusped hypocycloid $x^{2/3}+y^{2/3}=a^{2/3}$ show that

(i) $s=\frac{3}{4}a\cos 2\psi$, s being measured from the vertex; [*Nagpur*, 1954]

(ii) whole length of the curve is $6a$. [*Baroda*, 1960]

8. Prove that the cardioid $r=a(1+\cos\theta)$ is divided by the line $4r\cos\theta=3a$ into two parts such that the lengths of the arcs on either side of this line are equal. [*Nagpur*, 1960]

9. If s be the length of the curve

$$r=a\tanh\tfrac{1}{2}\theta$$

between the origin and $\theta=2\pi$, and Δ be the area under the curve between the same two points, prove that

$$\Delta=a(s-a\pi).$$ [*Panjab*, 1956]

10. Prove that the perimeter of the limaçon $r=a+b\cos\theta$, if b/a be small, is approximately

$$2\pi a(1+\tfrac{1}{4}b^2/a^2).$$ [*P.S.C., U.P.*, 1958]

11. Show that the whole length of the limaçon

$$r=a+b\cos\theta\quad(a>b)$$

is equal to that of an ellipse whose semi-axes are equal in

length to the maximum and minimum radii vectores of the limaçon. [*Banaras*, 1948]

[Hint. Take the ellipse to be

$$x=(a+b)\cos t,\ \ y=(a-b)\sin t.]$$

12. An ellipse of small eccentricity has its perimeter equal to that of a circle of radius a. Show that its area is

$$\pi a^2(1-\tfrac{3}{32}e^4) \text{ nearly.}$$

13. Show that the length of a loop of the curve

$$3x^2y-y^3=(x^2+y^2)^3,$$

is

$$2\int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^6)}}.$$

[Hint. Change the equation of the curve into polar coordinates. ζ is merely r.]

14. Trace roughly the curve $8a^2y^2=x^2(a^2-2x^2)$ and show that its whole length of arc is πa. [*Poona*, 1960]

Show that the area enclosed by the curve is $\frac{2}{3}$ of that of the circumscribing rectangle whose sides are parallel to the axes of coordinates.

15. Find the intrinsic equation of the spiral $r=a\theta$, the arc being measured from the pole.

16. Find the intrinsic equation to a parabola in its simplest form.

Deduce that the length of the arc intercepted between the vertex and an extremity of the latus rectum is

$$a\{\sqrt{2}+\log(1+\sqrt{2})\},$$

$4a$ being the latus rectum. [*Gorakhpur*, 1960]

17. Show that the intrinsic equation of the cycloid

$$x=a(t+\sin t),\ y=a(1-\cos t)$$

is

$$s=4a\sin\psi. \qquad [\textit{Agra}, 1963]$$

18. Show that the intrinsic equation of the semi-cubical parabola $3ay^2=2x^3$ is

$$9s=4a(\sec^3\psi-1).$$

19. Prove the formula

$$s=\int \frac{r\,dr}{\sqrt{(r^2-p^2)}}.$$

Show that the arc of the curve

$$p^2(a^4+r^4)=a^4r^2$$

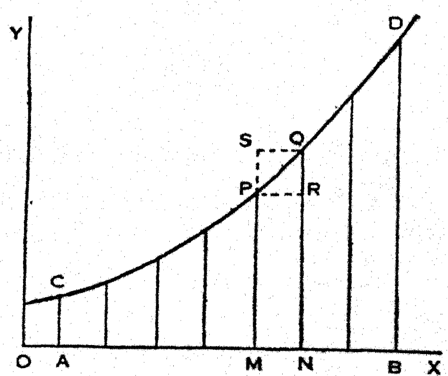
between the limits $r=b$, $r=c$ is equal in length to the arc of the hyperbola $xy=a^2$ between the limits $x=b$, $x=c$.

[*Delhi*, 1949]

CHAPTER VIII

VOLUMES AND SURFACES OF SOLIDS OF REVOLUTION

8·1. Volumes of solids of revolution. Let CD be the curve $y=f(x)$, and let a and b be the abscissae of C and D; and suppose that the volume of the solid generated by the revolution, about the x-axis, of the area bounded by the arc CD, the x-axis, and the ordinates of C and D is required.



Divide AB into n parts each equal to h, and erect ordinates at the points of division. Let the

ordinates at $x=a+rh$ and $x=a+(r+1)h$ be PM and QN. Construct as in the figure, and suppose* y goes on increasing as x increases from a to b.

We shall assume as an axiom that the volume of the solid generated by the revolution of the area $MNQP$ lies in magnitude between the volumes generated by the rectangles $MNRP$ and $MNQS$, i.e., between

$$\pi[f\{a+rh\}]^2h \quad \text{and} \quad \pi[f\{a+(r+1)h\}]^2h.$$

Adding up for each strip into which the area $ABDC$ has been divided, we see that the volume of the solid generated by the area $ABDC$ lies in magnitude between

$$\pi\sum_{r=0}^{n-1}[f\{a+rh\}]^2h \quad \text{and} \quad \pi\sum_{r=0}^{n-1}[f\{a+(r+1)h\}]^2h.$$

Now let h tend to 0. Then the two sums last written both tend to

$$\pi\int_a^b[f(x)]^2\,dx, \quad \text{i.e.,} \quad \pi\int_a^b y^2\,dx.$$

Hence the volume of the solid generated by the revolution, about the x-axis, of the area bounded by the curve $y=f(x)$, the ordinates at $x=a$, $x=b$, and the x-axis, is equal to

$$\pi\int_a^b \mathbf{y^2\,dx}.$$

This theorem can be easily modified to give the volume when the axis of revolution is not the x-axis, or when the equation is given in polar coordinates, or in a parametric form.

*This restriction can be easily removed as in the case of areas. See § 1·8.

Thus, interchanging x and y in the above formula, we see that the volume of the solid generated by revolving (about the y-axis) the area bounded by the curve, the lines $y=g$, $y=h$, and the y-axis, is equal to

$$\pi\int_g^h x^2\,dy,$$

in which the value of x in terms of y must be substituted from the equation of the curve.

If, however, the axis of revolution is not the y-axis, but a line (say $x=c$) parallel to it, the volume will be

$$\pi\int_g^h (x-c)^2\,dy,$$

for the perpendicular upon the axis of revolution from a point on the curve will now be $x-c$.

A similar formula can be written down when the axis of revolution is a line parallel to the x-axis.

If the curve is given by an equation in polar coordinates, say $r=f(\theta)$, and the curve revolves about the initial line, the volume generated

$$=\pi\int_a^b y^2\,dx=\pi\int_\alpha^\beta y^2\frac{dx}{d\theta}\,d\theta,$$

where α and β are the values of θ corresponding to the extremities C and D of the curve (for which $x=a$ and $x=b$ respectively).

Now $x=r\cos\theta$, $y=r\sin\theta$. Hence the volume

$$=\pi\int_\alpha^\beta r^2\sin^2\theta\,\frac{d}{d\theta}(r\cos\theta)\,d\theta,$$

in which the value of r in terms of θ must be substituted from the equation of the curve.

Similarly, if the curve is given by the equations

$$x=\phi(t),\quad y=\psi(t),$$

and if the extremities C and D of the curve correspond to the values k and l of t, the volume generated by the revolution of the area $ABDC$ about the x-axis

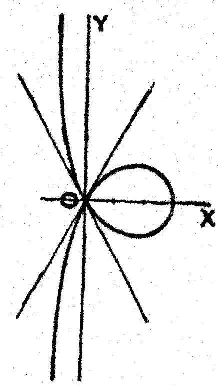
$$=\pi\int_k^l y^2\frac{dx}{dt}\,dt=\pi\int_k^l \{\psi(t)\}^2\,\phi'(t)\,dt.$$

Finally, if the curve CD revolves about some straight line AB which is not the x-or the y-axis, and if CA, DB are the perpendiculars on AB from C and D respectively, we can choose temporarily AB and AC as new axes of reference, say as the axes of ξ and η. Then the volume generated by the revolution of the area $ABDC$ about AB

$$=\pi\int_a^b \eta^2 \frac{d\xi}{dx}\,dx.$$

The integral can be evaluated by the usual methods after expressing η^2 and $d\xi/dx$ in terms of x, as in a worked out example below.

Ex. 1. The curve $y^2/(a+x)=x^2(3a-x)$ revolves about the axis of x. Find the volume generated by the loop.



Volume required $=\pi\int_0^{3a} y^2\,dx$

$$=\pi\int_0^{3a}\frac{x^2(3a-x)}{a+x}\,dx$$

$$=\pi\int_0^{3a}\left\{-x^2+4ax-4a^2+\frac{4a^3}{x+a}\right\}dx$$

$$=\pi\Big[-\tfrac{1}{3}x^3+2ax^2-4a^2x$$

$$+4a^3\log_e(x+a)\Big]_0^{3a}$$

$$=\pi(-3a^3+4a^3\log_e 4)=\pi(8\log_e 2-3)a^3.$$

Ex. 2. The part of the parabola $y^2=4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated.

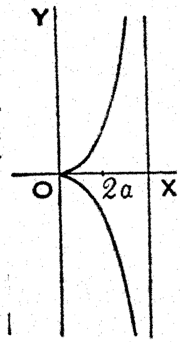
Here the axis of revolution is the y-axis and so we use the formula $\pi\int x^2\,dy$ instead of the formula $\pi\int y^2\,dx$. Also, on account of symmetry, we can find the volume of the solid generated by the parabola from the vertex to the end of the latus-rectum $(a, 2a)$ and double it to obtain the total volume. Thus the required volume

$$=2\pi\int_0^{2a} x^2\,dy=2\pi\int_0^{2a}\frac{y^4}{16a^2}\,dy=\frac{\pi}{8a^2}\left[\frac{y^5}{5}\right]_0^{2a}=\tfrac{4}{5}\pi a^3.$$

Ex. 3. Find the volume of the solid generated by the revolution of the cissoid

$$x=2a\sin^2 t,\ y=2a\sin^3 t/\cos t$$

about its asymptote.



The asymptote is $x=2a$. The perpendicular on it from any point (x, y) on the curve is, therefore, $2a-x$. Also, there is symmetry about the x-axis.

Hence the volume required

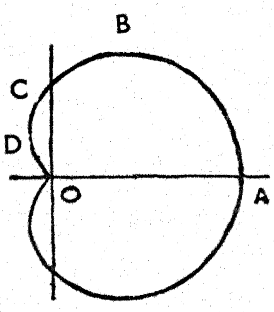
$$=2\pi\int_{y=0}^{\infty}(2a-x)^2\,dy=2\pi\int_{t=0}^{\pi/2}(2a-x)^2\frac{dy}{dt}\,dt$$

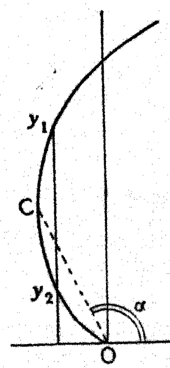
$$=2\pi\int_0^{\pi/2}4a^2\cos^4 t\,.\,2a.\frac{3\sin^2 t\cos^2 t+\sin^4 t}{\cos^2 t}\,dt$$

$$=16\pi a^3\int_0^{\pi/2}\cos^2 t\sin^2 t(1+2\cos^2 t)\,dt$$

$$=16\pi a^3\left\{\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2})}{2\Gamma(3)}+\frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{3}{2})}{\Gamma(4)}\right\}=2\pi^2 a^3.$$

Ex. 4. The cardioid $r=a(1+\cos\theta)$ revolves about the initial line. Find the volume of the solid generated. [*Del.*, '60]





The complete curve is as shown on the left. The portion in the third quadrant is shown on a larger scale on the right. We see that, although there is only one point on the curve for every value of θ, yet in the third quadrant there are two values of y (say y_1 and y_2) for every value of x. Let the point

on the curve which is on the extreme left hand side be C, and let its abscissa be $-k$ and the vectorial angle α. Then the volume required $=$the volume generated by the portion ABC of the curve $-$the volume generated by the portion CDO

$$=\pi\int_{-k}^{2a} y_1^2-\pi\int_{-k}^{0} y_2^2\,dx$$

$$=\pi\int_{0}^{-k} y_2^2\,dx+\pi\int_{-k}^{2a} y_1^2 dx, \text{ by § 5·2(ii)},$$

$$=\pi\int_{\theta=\pi}^{\theta=\alpha} y^2\frac{dx}{d\theta}\,d\theta+\pi\int_{\theta=\alpha}^{\theta=0} y^2\frac{dx}{d\theta}\,d\theta$$

$$=\pi\int_{\theta=\pi}^{\theta=0} y^2\cdot\frac{dx}{d\theta}\,d\theta$$

$$=-\pi\int_0^{\pi} a^2(1+\cos\theta)^2\sin^2\theta\,\frac{d}{d\theta}\{a(1+\cos\theta)\cos\theta\}\,d\theta$$

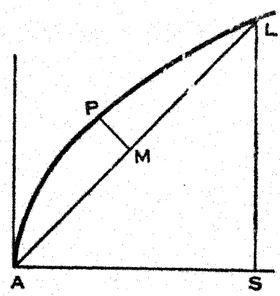
$$=\pi a^3\int_0^{\pi}(1+\cos\theta)^2\sin^3\theta\,(2\cos\theta+1)\,d\theta$$

$$=2\pi a^3\int_0^{\pi/2}(1+5\cos^2\theta)\sin^3\theta\,d\theta, \text{ by § 5·2},$$

$$=2\pi a^3\left\{\frac{\Gamma(2)\,\Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{2})}+5\frac{\Gamma(\frac{3}{2})\,\Gamma(2)}{2\Gamma(\frac{7}{2})}\right\}=\tfrac{8}{3}\pi a^3.$$

Ex. 5. The area cut off from the parabola $y^2=4ax$ by the chord joining the vertex to an end of the latus rectum is rotated through four right angles about the chord. Find the volume of the solid so formed. [*Agra*, 1960]

Let A be the vertex, S the focus, SL the latus rectum. Let P be any point (x, y) on the arc AL, and PM the perpendicular on AL.



Let $AM=\xi$, $PM=\eta$. Then, since the equation to AL is

$$y=2x,$$
$$\eta=PM=(y-2x)/\sqrt{5}$$
$$=2(\sqrt{a}\sqrt{x}-x)/\sqrt{5}.$$

Also $\quad AM^2=AP^2-PM^2$

$$=x^2+y^2-(y-2x)^2/5$$
$$=(5x^2+5y^2-y^2-4x^2+4xy)/5=(x+2y)^2/5.$$

Therefore $\xi=(x+4\sqrt{a}\sqrt{x})/\sqrt{5}$.

Hence the required volume

$$=\pi\int_0^{AL}\eta^2\,d\xi=\pi\int_0^a \eta^2\frac{d\xi}{dx}\,dx$$

$$=\pi\int_0^a \frac{4}{5\sqrt{5}}\,x(\sqrt{a}-\sqrt{x})^2\left\{1+\frac{2\sqrt{a}}{\sqrt{x}}\right\}dx.$$

Multiplying out and integrating, we easily find that the volume

$$=(2\sqrt{5})\pi a^3/75.$$

Examples

1. Show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. [*Lucknow*, 1951]

2. Find the volume of a spherical cap of height h cut off from a sphere of radius a. [*Baroda*, 1960]

3. A segment is cut off from a sphere of radius a by a plane at a distance $\frac{1}{2}a$ from the centre. Show that the volume of the segment is 5/32 of the volume of the sphere. [*Aligarh*, 1958]

4. Find the volume of the paraboloid generated by the revolution about the x-axis of the parabola $y^2=4ax$ from $x=0$ to $x=h$. [*Utkal*, 1956]

5. Find the volume of the solid generated by revolving the ellipse

$$x^2/a^2+y^2/b^2=1$$

about the x axis. [*Aligarh*, 1960]

6. The part of the ellipse

$$x^2/a^2+y^2/b^2=1$$

cut off by a latus rectum revolves about the tangent at the nearer vertex. Find the volume of the reel thus generated. [*Banaras, Geophysics*, 1957]

7. Prove that the volume of the solid generated by the revolution of an ellipse round its minor axis is a mean proportional between those generated by the revolution of the ellipse and of the auxiliary circle about the major axis. [*Panjab*, 1962]

8. If the hyperbola

$$x^2/a^2 - y^2/b^2 = 1$$

revolves about the x-axis, show that the volume included between the surface thus generated, the cone generated by the asymptotes and two planes perpendicular to the axis of x, at a distance h apart, is equal to that of a circular cylinder of height h and radius b.

9. A solid of height h is bounded by two parallel faces of areas A_1 and A_2, and the area of a parallel section at a distance x from one face is given by the formula ax^3+bx^2+cx+d; A is the area of the section which is exactly midway between the two faces. Show that the volume of the solid is

$$\tfrac{1}{6}h(A_1+4A+A_2). \qquad [U.P.F.S., 1953]$$

10. Prove that the volume of the solid generated by the revolution of the curve

$$y = \frac{a^3}{a^2+x^2}$$

about its asymptote is $\frac{1}{2}\pi^2 a^3$. [*Banaras, Geophysics*, 1960]

11. Trace roughly the curve $xy^2 = 4(2-x)$.

Find the area enclosed by the curve and y-axis, and also the volume of the solid formed by the revolution of the curve through four right angles about the y-axis.

12. Find the volume of the spindle-shaped solid generated by revolving the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ about the x-axis. [*Madras*, 1960]

8·2. Surface of solids of revolution. Let CD be the curve $y=f(x)$ (see the figure on p. 157) and let the abscissae of C and D be a and b. Further, let the length of the arc from C up to any point $P(x, y)$ be s, and suppose that the curved surface of the solid generated by the revolution of CD about the x-axis is required.

Divide AB into n parts each equal to h, and erect ordinates at the points of division. Let the

ordinates at $x=a+rh$ and $x=a+(r+1)h$ be PM and QN, and construct as in the figure. Let the arc PQ be equal to σ, and suppose* y goes on increasing as x increases from a to b.

We shall assume as an axiom that the curved surface of the solid generated by the revolution of $MNQP$ about the x-axis lies in magnitude between the curved surfaces of the two right circular cylinders, *each of thickness* σ, one of which has the radius PM and the other the radius QN, i.e., between

$$2\pi f\{a+rh\}\sigma \quad \text{and} \quad 2\pi f\{a+(r+1)h\}\sigma.$$

Adding up for each strip into which the area $ABDC$ has been divided, we see that the surface of the solid generated by the revolution of $ABDC$ lies in magnitude between

$$2\pi\sum_{r=0}^{n-1}\frac{\sigma}{h}f\{a+rh)h \text{ and } 2\pi\sum_{r=0}^{n-1}\frac{\sigma}{h}f\{a+(r+1)h\}h.$$

Now let h tend to zero. Then σ/h tends to ds/dx, and so the two sums last written both tend to

$$2\pi\int_a^b f(x)\frac{ds}{dx}\,dx, \quad \text{i.e.,} \quad 2\pi\int_{x=a}^{x=b} y\,ds.$$

Hence the curved surface of the solid generated by the revolution, about the x-axis, of the area bounded by the curve $y=f(x)$, the ordinates at $x=a$, $x=b$, and the x-axis, is equal to

$$2\pi\int_{x=a}^{x=b} \mathbf{y\,ds}.$$

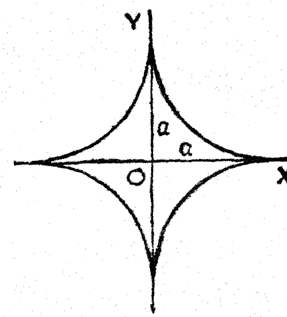
When the axis of revolution is not the x-axis, or when the equation is given in polar coordinates, the same devices may be used as for finding volumes.

*This restriction is easily removed as in § 1·8.

Ex. Find the surface of the solid generated by the revolution of the astroid

$$x=a\cos^3 t,\ y=a\sin^3 t$$

about the axis of x. [*Agra*, '62]



Here $dx/dt=-3a\cos^2 t\sin t$.

$dy/dt=3a\sin^2 t\cos t$.

Therefore $ds/dt=\pm 3a\sin t\cos t$.

Also, as x varies from 0 to a, t varies from $\frac{1}{2}\pi$ to 0; and there is symmetry about the y-axis.

Hence the required surface

$$=2\times 2\pi\int_{x=0}^{x=a} y\,ds=4\pi\int_{\pi/2}^{0} y\frac{ds}{dt}\,dt$$

$$=12\pi a^2\int_0^{\pi/2}\sin^4 t\cos t\,dt,$$ giving that sign to ds/dt which will give us a positive result,

$$=\tfrac{12}{5}\pi a^2\Big[\sin^5 t\Big]_0^{\pi/2}=\tfrac{12}{5}\pi a^2.$$

EXAMPLES

1. Show that the surface of the spherical zone, contained between two parallel planes is equal to $2\pi ah$, where a is the radius of the sphere and h the distance between the planes. [*Allahabad*, 1953]

2. A circular arc revolves about its chord. Prove that the area of the surface generated is $4\pi a^2(\sin\alpha-\alpha\cos\alpha)$, where a is the radius and 2α the angle subtended by the arc at the centre. [*Allahabad*, 1959]

3. Find the surface of the solid formed by the revolution, about the axis of y, of the part of the curve $ay^2=x^3$ from $x=0$ to $x=4a$ which is above the x-axis. [*Allahabad*, 1957]

4. The arc AL of a parabola, where A and L are respectively the vertex and an extremity of the latus rectum, is revolved about its axis. Find the area of the surface generated. [*Baroda*, 1960]

5. Find the area of the surface formed by the revolution of $x^2+4y^2=16$ about its major axis. [*Baroda*, 1960]

6. Prove that the surface of the prolate spheroid formed by the revolution of an ellipse of eccentricity e about its major axis is equal to

2 . area of ellipse . $\{\sqrt{(1-e^2)}+(\sin^{-1} e)/e\}$. [*I.A.S.*, 1952]

7. The coordinates of a point on a cycloid are

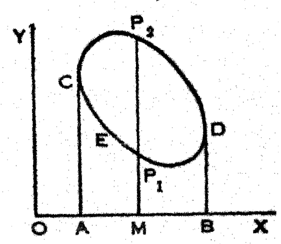
$$x=a(\theta+\sin\theta), \quad y=a(1+\cos\theta).$$

Show that the ratio of the area of the surface formed by the rotation of the arc of the cycloid between two consecutive cusps about the axis of x, to the area enclosed by the cycloid and the axis of x is $\frac{64}{9}$. [*Gorakhpur*, 1960]

8. Prove that the surface of any zone of a paraboloid of revolution is proportional to the difference of the radii of curvature of the generating curve at the points where it is cut by the bounding planes of the zones. [The solid formed by the revolution of a parabola about its axis is called a paraboloid of revolution.]

8·3. Theorems of Pappus*. (i) *If a closed curve revolves about a straight line in its plane, which does not intersect it, the* **volume** *of the ring thus formed is equal to the area of the curve multiplied by the length of the path of its centroid.*

Take the axis of rotation as the x-axis, and let CP_1DP_2 be the closed area.



Let CA and DB be the tangents parallel to the y-axis, their equations being $x=a$, $x=b$ respectively $(a<b)$.

Let the values of y corresponding to any x be y_1 and y_2.

*Pappus, a geometrician of great power, lived and taught at Alexandria about the end of the third century. The theorems about surfaces and volumes of solids of revolution now named after him were first given by him in his *Mathematical Collections*, the only work of his still extant. These theorems were re-discovered by P. Guldin over 1,000 years later, and so they are sometimes called *Guldin's Theorems.*

Then the volume generated by the closed area

$$=\text{volume generated by } ABDP_2C$$
$$-\text{volume generated by } ABDP_1C$$
$$=\pi\int_a^b y_2^2\,dx-\pi\int_a^b y_1^2\,dx=\pi\int_a^b (y_2^2-y_1^2)\,dx.$$

Now it is well known (or see next chapter) that the ordinate η of the centroid of the area of the closed curve CP_1DP_2 is given by

$$\eta=\frac{\frac{1}{2}\int_a^b (y_2^2-y_1^2)\,dx}{A},$$

where A is the area of the closed curve.

Hence the volume generated $=2\pi\eta A$, which proves the theorem.

(ii) *If an arc of a curve revolves about a straight line in its plane, which does not intersect it, the* **surface** *generated is equal to the* length *of the arc multiplied by the length of the centroid of the arc.*

Take the axis of rotation as the x-axis, and let the abscissae of the extremities of the arc be a and b. Then the surface generated by the revolution of the arc is equal to

$$2\pi\int_{x=a}^{x=b} y\,ds.$$

Now it is well known (or see next chapter) that the ordinate η_s of the centroid of the arc from $x=a$ to $x=b$, of length l, is given by

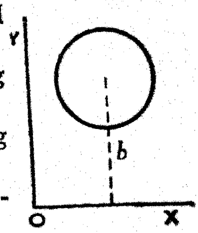
$$\eta_s=\frac{\int_{x=a}^{x=b} y\,ds}{l}.$$

Hence the surface generated $=2\pi\eta_s l$, which proves the theorem.

NOTE 1. The closed curve or arc in the above theorems must not cross the axis of revolution; but may be terminated by it, for even then the above proofs apply.

NOTE 2. When the volume or surface generated is otherwise known, the above theorems may be applied to determine the position of the centroid of the generating area or curve.

Ex. 1. Find the surface-area and volume of the anchor-ring generated by the revolution of a circle of radius a about an axis in its own plane distant b from its centre $(b>a)$. [*Raj.*, 1957]



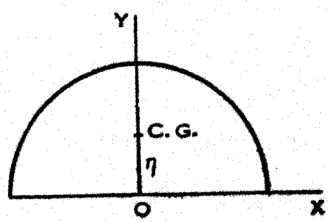
The centroid of the area of a circle and also of its circumference is the centre.

Hence the volume of the anchor-ring $=\pi a^2 \,.\, 2\pi b=2\pi^2 a^2 b$.

Again, the surface area of the anchor-ring $=2\pi a \,.\, 2\pi b=4\pi^2 ab$.

Ex. 2. Find the position of the centroid of a semi-circular area.

By symmetry it is evident that the centroid must be somewhere on the radius which is perpendicular to the bounding diameter. Let the distance of the centroid from the centre O be η.



Then $2\pi\eta\times$area of the semicircle$=$volume of a sphere of radius a.

Hence $$\eta=\frac{1}{2\pi}\cdot\frac{\frac{4}{3}\pi a^3}{\frac{1}{2}\pi a^2}=\frac{4a}{3\pi}.$$

EXAMPLES

1. Find the volume of the ring generated by the revolution of an ellipse of eccentricity $1/\sqrt{2}$ about a straight line parallel to the minor axis and situated at a distance from the centre equal to three times the major axis.

2. The loop of the curve $2ay^2=x(x-a)^2$ revolves about the straight line $y=a$; find the volume of the solid generated. [*Allahabad*, 1960]

3. A groove of semi-circular section of radius b is cut round a circular cylinder of radius a; prove that the volume removed is $\pi^2ab^2-\frac{4}{3}\pi b^3$. Show also that the area of the surface of the groove is $2\pi^2ab-4\pi b^2$.

4. Find the volume of the ring generated by the revolution of the cardioid $r=a(1+\cos\theta)$ about the line $r\cos\theta+a=0$, given that the centroid of the area of the cardioid is at a distance $\frac{5}{6}a$ from the origin.

5. Establish Pappus theorems on surfaces and volumes of solids of revolution.

Apply the results to determine the position of the centre of gravity of (i) a quadrant of a uniform circular lamina, (ii) a quadrant of a circular arc. [*Karnatak*, 1954]

6. The lemniscate $r^2=a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the volume generated is $\frac{1}{4}\pi^2a^3$. [*Panjab*, 1954]

EXAMPLES ON CHAPTER VIII

1. The hyperbola $x^2/a^2-y^2/b^2=1$ revolves about the axis of x. Show that the volume cut off from one of the two solids thus obtained by a plane perpendicular to the x-axis, and distant h from the vertex, is

$$\pi b^2h^2(3a+h)/3a^2.$$

2. The part of the curve $y^2=x^2(1-x^2)$ between $x=0$ and $x=1$ rotates about the x-axis. Obtain the volume of the solid thus generated.

3. Find the volume of the solid formed by the revolution of the loop of the curve $y^2=x^2(a-x)/(a+x)$ about the x-axis. [*Gorakhpur*, 1960]

4. Show that the volume of the solid generated by the revolution of the curve

$$(a-x)y^2=a^2x$$

about its asymptote is $\frac{1}{2}\pi^2a^3$. [*Allahabad*, 1959]

5. Find the volume generated by the revolution of the loop of the curve $y^2=x^4(x+2)$ about the axis of x.

6. A basin is formed by the revolution of the curve $x^3=64y$, $(y>0)$ about the axis of y. If the depth of the basin is 8 inches, how many cubic inches of water will it hold ? [*Baroda*, 1959]

7. A quadrant of a circle of radius a revolves about its chord. Show that the volume of the spindle generated is

$$(\pi/6\sqrt{2})(10-3\pi)a^3. \qquad [\textit{Nagpur}, 1952]$$

8. Find the volume of the solid generated by rotating completely about the x-axis the area enclosed between $y^2=x^3+5x$ and the lines $x=2$ and $x=4$.

9. The volume of a hemisphere is divided into two equal parts by a plane parallel to its base. Show that the distance of the plane from the base lies between three-tenths and four-tenths of the radius of the hemisphere. [*U.P.F.S.*, '53]

10. The figure bounded by a quadrant of a circle of radius a and the tangents at its extremities revolves about one of the tangents. Prove that the volume of the solid generated is $(\frac{5}{3}-\frac{1}{2}\pi)\pi a^3$. [*Poona*, 1956]

11. The area between a parabola and its latus rectum revolves about the directrix. Find the ratio of the volume of the ring thus obtained to the volume of the sphere whose diameter is the latus rectum. [*Allahabad*, 1958]

12. If b be the radius of the middle section of a cask, and a the radius of either end, prove that the volume of the cask is

$$\tfrac{1}{15}\pi(3a^2+4ab+8b^2)h,$$

where h is the length of the cask, it being assumed that the generating curve is an arc of a parabola.

13. Find the volumes of the oblate and prolate spheroids generated by an ellipse whose major and minor axes are

$$(24\pi)^{1/3} \text{ and } (3\pi)^{1/3}. \qquad [\textit{Nagpur}, 1956]$$

14. The ellipse $b^2x^2+a^2y^2=a^2b^2$ is divided into two parts by the line $x=\frac{1}{2}a$, and the smaller part is rotated through four right angles about this line. Prove that the volume generated is

$$\pi a^2b(\tfrac{3}{4}\sqrt{3}-\tfrac{1}{3}\pi). \qquad [\textit{Rajasthan}, 1960]$$

15. A solid spheroid formed by the revolution of the ellipse $b^2x^2+a^2y^2=a^2b^2$ about the major axis has a cylindrical hole of circular section having the major axis as axis drilled through it. Prove that the volume of the solid which remains is $4\pi b^2l^3/3a^2$ where $2l$ is the length of the hole. [*Gorakhpur*, 1960]

16. Find the area included between the curves $y^2=x^3$ and $x^2=y^3$, and find the volume of the solid of revolution obtained by rotating this area about the x-axis. [*Nagpur*, '53]

17. Show that if the area lying within the cardioid

$$r=2a(1+\cos\theta)$$

and without the parabola $r(1+\cos\theta)=2a$ revolves about the initial line, the volume generated is $18\pi a^3$. [*Ujjain*, 1960]

18. Show that the volume of the solid generated by the revolution of the cycloid

$$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta),\ 0\leqslant\theta\leqslant\pi,$$

about the y-axis is $\pi a^3(\frac{3}{2}\pi^2-\frac{8}{3})$. [*Rajasthan*, 1961]

19. Prove that the volume of the reel formed by the revolution of the cycloid

$$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta)$$

about the tangent at the vertex is π^2a^3. [*P.C.S., U.P., Forest*'60]

20. Find the area A between the curve

$$y=a(\sin x+\tfrac{1}{3}\sin 3x+\tfrac{1}{5}\sin 5x)$$

and the axis of x between the termini (limits) 0 and π; and the volume V obtained by rotating this area about the axis of x. Prove that $4V=\pi^2aA$. [*Madras*, 1951]

21. Sketch the curves

$$xy^2=a^2(a-x),\ (a-x)y^2=a^2x.$$

Prove that the volume obtained by revolution about $x=\frac{1}{2}a$ of the area enclosed between these curves is $\pi a^3(4-\pi)/4$. [*Banaras*, 1956]

22. Find the volume of the solid formed by revolving one loop of the curve

$$r^2=a^2\cos 2\theta.$$

(i) about the initial line, and (ii) about the line $\theta=\pi/2$.
[*First part, Rajasthan*, 1962; *Second part, Rajasthan* 1960]

23. Show that the curve $r=1+2\cos\theta$ consists of an outer and an inner loop.

If the area of the inner loop is rotated through two right angles about the initial line, show that the volume of the solid so formed is $\pi/12$. [*Delhi*, 1962]

24. Prove that the volume generated by the revolution of the limaçon $r=a+b\cos\theta$, $a>b$, about the initial line, is

$$\tfrac{4}{3}\pi a(a^2+b^2). \qquad \text{[\textit{Gorakhpur}, 1959]}$$

25. Prove that the volume of the solid generated by the revolution of the conchoid $r=a+b\sec\theta$ $(-\frac{1}{2}\pi<\theta<\frac{1}{2}\pi)$ about its asymptote is

$$2\pi a^2(\tfrac{2}{3}a+\tfrac{1}{2}\pi b).$$

26. The part of the parabola $y^2=4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus generated. [*Rajasthan*, 1960]

27. Find the area of the surface swept out by the arc of the rectangular hyperbola $x^2-y^2=a^2$, extending from the vertex to the end of the latus rectum, when rotated through four right angles about the axis of x. [*Allahabad*, 1957]

28. The arc of the cardioid $r=a(1+\cos\theta)$ included between $-\frac{1}{2}\pi\leqslant\theta\leqslant\frac{1}{2}\pi$ is rotated about the line $\theta=\frac{1}{2}\pi$. Find the area of the surface generated. [*Nagpur*,' 1961]

29. The curve

$$r=a(1+\cos\theta)$$

revolves about the initial line. Find the volume and the surface of the figure formed. [*First Part, Rajasthan*,' 61]

30. The lemniscate $r^2=a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the surface of the solid generated is $4\pi a^2$. [*Delhi Hons.*, 1959]

31. Prove that the surface and volume of the solid generated by the revolution, about the x-axis, of the loop of the curve

$$x=t^2, \quad y=t-\tfrac{1}{3}t^3$$

are respectively 3π and $\frac{3}{4}\pi$. [*Delhi Hons.*, 1958]

32. Prove that the surface and volume generated by the revolution of the tractrix

$$x=a\cos t+\tfrac{1}{2}a\log\tan^2\tfrac{1}{2}t,$$
$$y=a\sin t,$$

about its asymptote are respectively equal to the surface and half the volume of a sphere of radius a.

[*Agra*, 1959; *First part*, *Ujjain*, 1960]

33. Find the volume and also the surface generated by the revolution of the catenary

$$y=c\cosh(x/c)$$

about the axis of x. [*Nagpur*, 1951]

34. $A(0, a)$ and $P(x, y)$ are two points on the curve whose equation is $y=a\cosh(x/a)$, and s is the length of the arc AP. If the curve makes a complete revolution about the x-axis, prove that the area S of the curved surface, bounded by planes through A and P perpendicular to the x-axis, and the corresponding volume V are connected by

$$aS=2V=\pi a(ax+sy).$$

[*P.S.C.*, *U.P.*, *Forest*, 1959]

35. An area lies altogether on one side of an axis in its plane. Prove that the volume of the solid formed by the rotation of the area about the axis is equal to the area multiplied by the distance traversed by its centre of gravity.

Hence prove that the volume of the solid formed by the rotation about the line $\theta=0$ of the area bounded by the curve $r=f(\theta)$ and the lines $\theta=\theta_1$, $\theta=\theta_2$, is

$$\tfrac{2}{3}\pi\int_{\theta_1}^{\theta_2} r^3\sin\theta\,d\theta.$$

[*Sagar*, 1950]

Find the volume of the solid formed by the revolution of the cardioid $r=a(1+\cos\theta)$ about the line $\theta=0$.

CHAPTER IX

APPLICATIONS

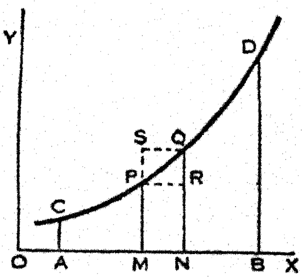
9·1. Centre of gravity. It is proved in books on Statics that if the centre of gravity of masses m_1, m_2, ..., m_n, which have their centres of gravity at (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), is (ξ, η), then

$$\xi = \frac{\Sigma mx}{\Sigma m}, \quad \eta = \frac{\Sigma my}{\Sigma m},$$

where Σmx means $m_1 x_1 + m_2 x_2 + \ldots + m_n x_n$, and Σm and Σmy have similar meanings.

In the case of continuous distribution of matter, the above summations become definite integrals.

(i) *Centre of gravity of an arc.*



Let CD be an arc of the curve $y=f(x)$ from $x=a$ to $x=b$, and let CA and DB be the ordinates of C and D. Divide AB into n parts, each of length h, and erect ordinates at the points of division. Let the points the abscissae of which are $a+rh$ and $a+(r+1)h$ be P and Q, and let σ be the length of the arc PQ and s that of CP.

Then, if λ be the mass of unit length of the arc, it is obvious that Σmx for the arc PQ lies in

magnitude between $\lambda\sigma . OM$ and $\lambda\sigma . ON$, i.e., between

$$\lambda\sigma\{a+rh\} \text{ and } \lambda\sigma\{a+(r+1)h\}.$$

Adding up for all the parts into which CD has been divided, we find that Σmx for the arc CD lies between

$$\sum_{r=0}^{n-1} \lambda(\sigma/h)\{a+rh\}h \text{ and } \sum_{r=0}^{n-1} \lambda(\sigma/h)\{a+(r+1)h\}h. \quad . \quad . \quad . \quad (1)$$

Now let h tend to zero. Then the two sums last written both tend to

$$\lambda\int_{x=a}^{x=b} x\,\frac{ds}{dx}\,dx, \text{ i.e., } \lambda\int_{x=a}^{x=b} x\,ds,$$

supposing that λ is a constant.

Also Σm=mass of the arc $CD=\lambda L$ if the length of the arc CD is L.

Hence $$\xi=\frac{\int_{x=a}^{x=b}\mathbf{x\,ds}}{\mathbf{L}}.$$

Similarly $$\eta=\frac{\int_{x=a}^{x=b}\mathbf{y\,ds}}{\mathbf{L}},$$

the only modification required in the above proof being that in the sums (1) there will be

$$f\{a+rh\} \text{ and } f\{a+(r+1)h\}$$

instead of $a+rh$ and $a+(r+1)h$, so that we shall get y in the final result instead of x.

(ii) *Centre of gravity of an area.* Suppose the centre of gravity (ξ, η) of the area bounded by the

curve CD, the x-axis, and the ordinates at C and D is required. Let μ be the mass per unit area.

We shall find η first.

With the same construction as before, the area under consideration is divided into strips. The contribution to Σmy by any one strip of breadth h is equal to

$\mu \times$ area of the strip $MNQP \times$ distance of its centre of gravity from the x-axis.

Now for the rectangle $MNRP$ the mass is evidently less and its centre of gravity is nearer to the x-axis than for the strip $MNQP$. The reverse is the case for the rectangle $MNQS$.

Also, the distance of the centre of gravity of the rectangle $MNRP$ or $MNQS$, from the x-axis, is equal to half its height.

Hence Σmy for the strip $MNQP$ lies in magnitude between

$$\mu \,.\, hf\{a+rh\} \,.\, \tfrac{1}{2}f\{a+rh\}$$

and $$\mu \,.\, hf\{a+(r+1)h\} \,.\, \tfrac{1}{2}f\{a+(r+1)h\}.$$

Summing up for all the strips, and assuming that μ is a constant, we see that Σmy for the area $ABDC$ lies in magnitude between

$$\tfrac{1}{2}\mu\sum_{r=0}^{n-1}[f\{a+rh\}]^2h \text{ and } \tfrac{1}{2}\mu\sum_{r=0}^{n-1}[f\{a+(r+1)h\}]^2h.$$

Now let n tend to infinity. Then both these sums tend to

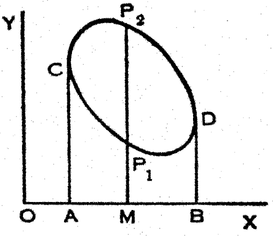
$$\tfrac{1}{2}\mu\int_{x=a}^{x=b}[f(x)]^2\,dx.$$

Also $\Sigma m = \mu\alpha$, where α is the area of the figure *ABDC*.

Hence $$\eta = \frac{\frac{1}{2}\int_a^b \mathbf{y}^2\,\mathbf{dx}}{\alpha}.$$

Similarly $$\xi = \frac{\int_a^b \mathbf{xy}\,\mathbf{dx}}{\alpha}.$$

(iii) *Centre of gravity of a closed areanot cutting the axes*. Suppose that the centre of gravity (ξ, η) of the closed area CP_1DP_2 is required.



Let CA, DB be the tangents parallel to the y-axis, A and B being on the x-axis. Let $OA = a$, $OB = b$. Let the values of y corresponding to any x be y_1 and y_2.

Then the figure CP_1DP_2, of area α, may be regarded as the difference of the figures $ABDP_2C$ and $ABDP_1C$, of areas, say α_2 and α_1, and with centres of gravity at distances η_2 and η_1 from the x-axis. Then, if μ be the mass per unit area, the fundamental formula for the position of the centre of gravity gives

$$\eta = \frac{\mu\alpha_2\eta_2 - \mu\alpha_1\eta_1}{\mu\alpha_2 - \mu\alpha_1}$$

$$= \frac{\frac{1}{2}\int_a^b y_2^{\,2}\,dx - \frac{1}{2}\int_a^b y_1^{\,2}\,dx}{\alpha_2 - \alpha_1}$$

$$=\frac{\frac{1}{2}\int_a^b (y_2^2-y_1^2)\,dx}{a}.$$

(iv) *Centre of gravity of a surface of revolution.* By methods similar to those used above, it is easy to show that the centre of gravity of the surface, of area S, generated by the revolution of the curve $y=f(x)$ from $x=a$ to $x=b$, is $(\xi, 0)$ where

$$\xi=\frac{2\pi\int_{x=a}^{x=b} \mathbf{xy\,ds}}{\mathbf{S}}.$$

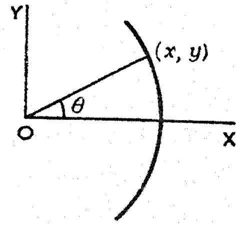
(v) *Centre of gravity of a solid of revolution.* We can also show similarly that the centre of gravity of the solid, of volume V, generated by the revolution of the area bounded by the curve $y=f(x)$, the ordinates at $x=a$ and $x=b$, and the x-axis, is $(\xi, 0)$, where

$$\xi=\frac{\pi\int_a^b \mathbf{xy^2\,dx}}{\mathbf{V}}.$$

NOTE 1. The *centroid*, *centre of mass*, *centre of inertia*, *centre of position* all coincide with the centre of gravity.

2. If λ, μ, or ρ (volume-density) is variable, the above formulae all become modified. It is easy to see that the new denominators in the expressions for ξ and η will be the mass of the arc, area, surface or solid whose centre of gravity is to be determined, and the integrands in the new numerators will be the old integrands multiplied by the factor λ, μ, or ρ as the case may be, it being assumed in the last case that ρ is a function of x alone.

Ex. 1. Find the position of the centre of gravity of an arc of a circle of radius a, which subtends an angle 2α at the centre.



Take the centre of the circle as the origin and the radius which bisects the arc as the x-axis

If (ξ, η) be the centre of gravity, it is evident by symmetry that $\eta=0$. Also, if the vectorial angle of any point (x, y) on the arc is θ, we have by section (i) above, since the length of the arc is equal to $2a\alpha$,

$$2a\alpha\,\xi=\int_{\theta=-\alpha}^{\alpha} x\,ds=\int_{-\alpha}^{\alpha} a\cos\theta\,.\,a\,d\theta=2a^2\int_{0}^{\alpha}\cos\theta\,d\theta$$
$$=2a^2\sin\alpha.$$

Hence $$\xi=\frac{a\sin\alpha}{\alpha}.$$

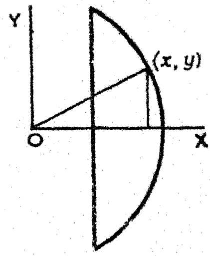
Ex. 2. Find the centroid of the area enclosed by the parabola $y^2=4ax$ and the double ordinate $x=h$. [*Utkal*, 1951]

By symmetry, if (ξ, η) be the centre of gravity, $\eta=0$. Also the abscissa of the centre of gravity will be the same whether we consider the area on both sides of the x-axis, or the area on one side only. But if we consider the area on one side only, the formula of section (ii) above will apply. Hence

$$\xi=\frac{\int_0^h xy\,dx}{\int_0^h y\,dx}=\frac{\int_0^h x^{3/2}\,dx}{\int_0^h x^{1/2}\,dx}=\frac{\frac{2}{5}h^{5/2}}{\frac{2}{3}h^{3/2}}=\tfrac{3}{5}h.$$

Ex. 3. Find the centre of gravity of the segment of a sphere of radius a, cut off by a plane at a distance h from the centre. Hence deduce the position of the centre of gravity of a hemisphere. [*Karnatak*, 1959]

Take the centre of the sphere as the origin and the radius perpendicular to the plane base of the segment as the axis of x. Then, by section (v) above,



$$\xi=\frac{\pi\int_h^a xy^2\,dx}{\pi\int_h^a y^2\,dx}=\frac{\int_h^a x(a^2-x^2)\,dx}{\int_h^a (a^2-x^2)\,dx}$$

$$=\left[\tfrac{1}{2}a^2x^2-\tfrac{1}{4}x^4\right]_h^a\div\left[a^2x-\tfrac{1}{3}x^3\right]_h^a$$

$$=\frac{\frac{1}{4}a^4-\frac{1}{2}a^2h^2+\frac{1}{4}h^4}{\frac{2}{3}a^3-a^2h+\frac{1}{3}h^3}=\frac{3(a^2-h^2)^2}{4(2a^3-3a^2h+h^3)}$$

$$=\frac{3(a-h)^2(a+h)^2}{4(a-h)^2(2a+h)}=\frac{3(a+h)^2}{4(2a+h)}.$$

Putting $h=0$, we see that the C.G. of a hemisphere is at a distance $\frac{3}{8}a$ from the plane base.

EXAMPLES*

Find the centre of gravity of

1. The arc of the parabola $y^2=4ax$ included between the vertex and the point whose abscissa is at^2.

2. The arc of the catenary $y=a\cosh(x/a)$ from the vertex to the point (x, y).

3. The arc of the curve $x^{2/3}+y^{2/3}=a^{2/3}$ included between two successive cusps.

4. A sector of a circle.

5. A segment of a circle; in particular, a semi-circle.

6. The area between the curve $y=\sin x$ from $x=0$ to $x=\pi$ and the x-axis.

7. The area between $ay^2=x^3$, the x-axis and the ordinate at $x=b$. [*Aligarh*, 1949]

8. The area between the curve $y^2(2a-x)=x^3$ and its asymptote. [*Aligarh*, 1948]

*From Loney's *Elementary Treatise on Statics*.

9. The area of the loop of the curve

$$y^2(a+x)=x^2(a-x).$$

10. The area within the cardioid $r=a(1+\cos\theta)$. [*Baroda*, 1959]

11. The area of one loop of the lemniscate

$$r^2=a^2\cos 2\theta.$$

12. The area enclosed by the curves $y^2=ax$ and $x^2=by$. [*Banaras, Eng.*, 1958]

13. The area cut off from the parabola $y^2=4ax$ by the straight line $y=mx$.

14. The surface formed by the revolution of the cardioid $r=a(1+\cos\theta)$ about its axis.

15. The solid formed by the revolution about the x-axis, of the parabola $y^2=4ax$ cut off by the ordinate $x=h$. [*Andhra*, 1954]

16. The solid formed by the revolution of the cardioid

$$r=a(1+\cos\theta)$$

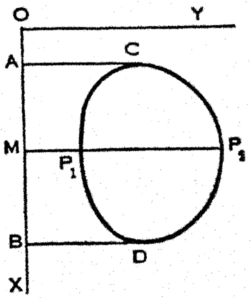
about the initial line.

9·2. Centre of pressure. If a plane area be in contact with a liquid, the point in the plane area at which the resultant pressure acts is called the *centre of pressure* of the area. We shall suppose throughout that the plane area is vertical.

It is shown in books on Hydrostatics that if the plane area be divided into a number of parts, and if α be the area and x the depth (below the surface of the liquid) of the centre of pressure of any such part, then the depth of the centre of pressure of the complete area is ξ, where

$$\xi=(\Sigma\alpha x^2)/\Sigma\alpha x.$$

Let CP_1DP_2 be the plane area, CA and DB the horizontal tangents, A and B being on the x-axis. Let $OA=a$, $OB=b$. Divide AB into n parts, each of length h, and draw horizontal lines through the points of division. Taking axes as in the figure, let y_1 and y_2 be the values of the ordinates corresponding to any value x of the abscissa.



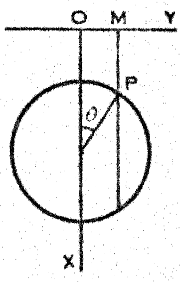
Then we can show, exactly as in the case of the centre of gravity, that

$$\xi=\left\{\int_a^b (y_2-y_1)x^2\,dx\right\}\bigg/\int_a^b (y_2-y_1)x\,dx;$$

also, that the ordinate η of the centre of pressure is given by

$$\eta=\left\{\tfrac{1}{2}\int_a^b (y_2{}^2-y_1{}^2)x\,dx\right\}\bigg/\int_a^b (y_2-y_1)x\,dx.$$

Ex. A circle of radius a is immersed vertically in a liquid, the depth of the centre of the circle below the surface of the liquid being h; find the depth of the centre of pressure.



Take axes as in the figure and let θ be the angle which the radius to any point P, (x_1, y_1), on it makes with the negative direction of the x-axis. Then $\eta=0$ by symmetry. Also, if the values of y corresponding to x are y_1 and y_2, then $y_2-y_1=2a\sin\theta$, $x=h-a\cos\theta$, $dx/d\theta=a\sin\theta$. Hence

$$\xi=\left\{\int_{h-a}^{h+a} 2a(\sin\theta)\,x^2\,dx\right\}\bigg/\int_{h-a}^{h+a} 2a(\sin\theta)\,x\,dx$$

$$=\left\{\int_0^\pi \sin^2\theta(h-a\cos\theta)^2\,d\theta\right\}\Big/\int_0^\pi \sin^2\theta\,(h-a\cos\theta)\,d\theta,$$

since $x=h-a\cos\theta$,

$$=\left\{2\int_0^{\pi/2} \sin^2\theta\,(h^2+a^2\cos^2\theta)\,d\theta\right\}\Big/2\int_0^{\pi/2} h\sin^2\theta\,d\theta,$$

by § 5·2,

$$=\frac{h^2\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})/2\Gamma(2)+a^2\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})/2\Gamma(3)}{h\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})/2\Gamma(2)}$$

$$=h+a^2/4h.$$

Examples

Find the centre of pressure of the following area when immersed vertically in a liquid :

1. A rectangle with one side in the surface of the liquid.

2. A triangle with its base in the surface.

3. A triangle with its vertex in the surface of the liquid and base horizontal.

4. A rectangle with two sides horizontal and at depths h_1 and h_2 below the surface.

5. A triangle with one side horizontal and the vertices at depths h_1, h_2 and h_2 $(h_1<h_2)$.

6. A semicircular area, when the radius is a, and the depth of the bounding diameter (which is horizontal and nearest the surface) is b.

7. An ellipse, completely immersed, with the minor axis horizontal and at depth h.

8. A completely immersed segment of a parabola bounded by the latus rectum with the axis vertical and the vertex downwards and at a depth h.

9. An area bounded by the curve $ay^2=x^3$, an abscissa of length h and the ordinate at its extremity, is

placed in water with this ordinate in the surface. Prove that the depth of the centre of pressure is $\frac{3}{8}h$.

10. A square is immersed with its diagonal vertical and its lowest point as deep again as its highest point. Find the depth of its centre of pressure.

9·3. Moment of inertia. If particles of masses $m_1, m_2, .., m_n$ be situated at points whose perpendicular distances from a given straight line are $r_1, r_2, ..., r_n$, then

$$\Sigma mr^2,$$

i.e., $m_1r_1^2+m_2r_2^2+...+m_nr_n^2$ is called the *moment of inertia* of the system about the given line.

The moment of inertia is of great importance in the dynamics of rigid bodies. Thus the kinetic energy of a body rotating with angular velocity ω about an axis AB is equal to

$$\tfrac{1}{2}(\text{moment of inertia of the body about } AB)\times\omega^2.$$

If the moment of inertia of a body of mass M about any axis AB be Mk^2, then k is called the *radius of gyration* of the body about AB.

The moment of inertia of a single particle of mass m and at a distance r from the given line is thus mr^2 about the given line. If instead of a single particle, we have a straight or a circular line, or a cylindrical surface, whose mass is m and every point of which is at the same distance r from the given line, then the moment of inertia about the given line of the mass m will evidently be mr^2.

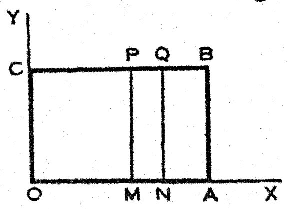
These considerations and the following two theorems enable us to find the moments of inertia of several bodies.

1. *Let GX be any straight line through the centre of inertia G of a body of mass M, and let OX′ be any parallel straight line. Then the moment of inertia of the body about OX′ is equal to the moment of inertia of the body about GX together with the moment of inertia of a particle of mass M, placed at G, about OX′.*

2. *The moment of inertia of a plane lamina about any straight line OZ perpendicular to it is equal to the sum of the moments of inertia about any two perpendicular straight lines OX, OY in the lamina which pass through the point of intersection O of the lamina and OZ.*

The proofs of these theorems do not depend upon integration and will be found in any text-book on Rigid Dynamics.

Ex. 1. Find the moment of inertia of a rectangle about one side and deduce the moment of inertia of a thin rod of length $2a$ about an axis through the middle-point perpendicular to the rod.



Take two adjacent sides of the rectangle as the axes of reference as in the figure.

Let the sides OA, OC be of lengths a and b respectively, and let μ be the surface density.

Divide OA into n parts each equal to h.

Let $OM=rh$, $MN=h$.

The mass of $MNQP=\mu \,.\, MN \,.\, MP=\mu bh$.

Then the moment of inertia of $MNQP$ about OC lies between

$$\mu bh(rh)^2 \quad \text{and} \quad \mu bh\{(r+1)h\}^2.$$

Summing up and taking limits, we see that the moment of inertia of the rectangle $OABC$ about OC is equal to

$$b\int_0^a \mu x^2\,dx.$$

If μ is a constant, we see that *the moment of inertia of a rectangle, one of whose sides is of length a, about the other side is equal to* $\frac{1}{3}\mu a^3 b$, i.e.,

$$\mathbf{Ma^2/3},$$

where M is the mass of the rectangle.

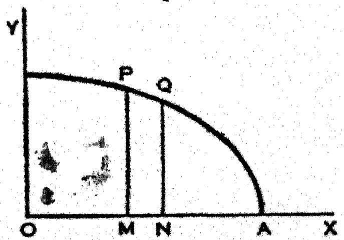
Corollary. It follows from the above that *the moment of inertia of a rectangle of length 2a about a straight line through its centre bisecting the sides of length 2a is* $\mathbf{Ma^2/3}$. For, the rectangle of length $2a$ can be regarded as composed of two rectangles each of length a.

Consequently *the moment of inertia of a thin rod of length 2a about an axis through its centre perpendicular to the rod is* $\mathbf{Ma^2/3}$. For, we can regard the thin rod as a thin rectangle.

These results can be also easily established independently.

Note. By applying Theorem 2, we see that *the moment of inertia of a rectangle having sides of lengths 2a, 2b, about an axis through its centre perpendicular to its plane is* $\frac{1}{3}\mathbf{M(a^2+b^2)}$.

It follows from this that *the moment of inertia of a rectangular parallelopiped having sides of lengths 2a, 2b, 2c, about an axis through the centre parallel to the side 2c is* $\frac{1}{3}\mathbf{M(a^2+b^2)}$. For, we can regard the parallelopiped as made up of an infinite number of thin rectangles.



Ex. 2. Find the moment of inertia of an elliptic disc having axes of lengths $2a$, $2b$, about the major axis.

Take the major axis as the axis of x, and the minor axis as the axis of y.

Divide OA into n parts, each equal to h, and erect ordinates at the points of division. Let PM and QN be the ordinates at distances rh and $(r+1)h$ from O. Let μ be the surface-density.

Then, by Ex. 1, the moment of inertia of $MNQP$ about OX lies between $\mu . PM . h . \frac{1}{3}PM^2$ and $\mu . QN . h . \frac{1}{3}QN^2$. Summing up and taking limits, and noting that the ellipse is symmetrical about both axes, we see that the moment of inertia of the complete ellipse about OX is equal to

$$4\int_0^a \mu . \tfrac{1}{3}y^3 \, dx.$$

Suppose μ is constant. Also let $x=a\cos\phi$. Then $y=b\sin\phi$, and the moment of inertia required

$$=-\tfrac{4}{3}\mu ab^3\int_{\pi/2}^{0}\sin^4\phi \, d\phi=\tfrac{4}{3}\mu ab^3\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{2\Gamma(3)}$$

$$=\tfrac{4}{3} . \tfrac{3}{2} . \tfrac{1}{2}\pi . \tfrac{1}{4}\mu ab^3,$$

i.e., *the moment of inertia of an ellipse about the major axis is* $\mathbf{Mb^2/4}$, where $M(=\pi ab\mu)$ is the mass of the ellipse.

Similarly, the moment of inertia about the minor axis is $Ma^2/4$.

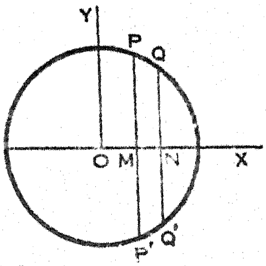
COROLLARIES. It follows that *the moment of inertia of a circle of radius* **a** *about any diameter is* $\mathbf{Ma^2/4}$, *and about an axis through the centre perpendicular to its plane is* $\mathbf{Ma^2/2}$. Evidently this must also be the moment of inertia of a right circular cylinder of mass M and radius a about its axis.

These results can be easily established independently also.

Ex. 3. Find the moment of inertia of a solid sphere about a diameter. [*Utkal*, 1949]

Take the diameter as the axis of x and the centre as the origin. Divide the radius along the x-axis (of length a, say) into n parts each equal to h, and draw through the points of division planes perpendicular to the radius.

If the plane PP' is at the distance rh from the origin and QQ' at the distance $(r+1)h$, the moment of inertia



of the disc $P'Q'QP$ evidently lies between the moments of inertia of the two right circular cylinders each of height h, but of radii PM and QN respectively, i.e. lies (by the corollary to Ex. 2) between

$$\rho \cdot \pi \cdot PM^2 \cdot h \cdot \tfrac{1}{2} PM^2$$

and $$\rho \cdot \pi \cdot QN^2 \cdot h \cdot \tfrac{1}{2} QN^2,$$

where ρ is the density of the sphere.

Summing up and taking limits, we see that the moment of inertia of the sphere

$$=\tfrac{1}{2}\pi\int_{-a}^{a} \rho y^4\, dx=\tfrac{1}{2}\pi\rho\int_{-a}^{a} (a^2-x^2)^2\, dx, \text{ if } \rho \text{ is constant,}$$

$$=\pi\rho\int_{0}^{a}(a^4-2a^2x^2+x^4)\, dx=\pi\rho a^5(1-\tfrac{2}{3}+\tfrac{1}{5})$$

$$=\tfrac{4}{3}\pi a^3\rho \cdot \tfrac{3}{4} \cdot \tfrac{8}{15}a^2,$$

i.e., *the moment of inertia of a sphere about a diameter*

$$=\mathbf{M \cdot 2a^2/5.}$$

Corollary. By differentiation of the value $\frac{2}{5} \cdot \frac{4}{3}\pi a^5\rho$ of the moment of inertia of a solid sphere, we see that *the moment of inertia of a thin hollow sphere (spherical shell) of radius a is*

$$\tfrac{8}{3}\pi a^4\rho \cdot (\text{thickness of shell}),$$

i.e., $$\mathbf{M \cdot 2a^2/3.}$$

Examples

Find the moment of inertia of :

1. A spheroid about its axis of revolution.

2. The paraboloid generated by the revolution of the parabola $y^2=4ax$ about the x-axis from $x=0$ to $x=h$.

3. A thin uniform circular ring about a diameter. [*Utkal*, 1956]

4. A portion of a uniform thin circular ring about the straight line joining its extremities.

5. The area bounded by one arch of a cycloid and the base about the base.

6. A thin rod of which the line density varies as the distance from one end about an axis passing through that end and at right angles to the rod.

9·4. Other applications. The definite integral can represent many important magnitudes in physics. In particular, if $f(x)$ is a force acting along the x-axis, whose magnitude is a function $f(x)$ of the distance x of its point of application $(x, 0)$ from the origin, the work done as the particle moves from a to b is

$$\int_a^b f(x)\,dx.$$

Similarly
$$\int_a^b p\,dv$$
represents the work done as a gas expands from the volume a to the volume b, the pressure for any volume v being p.

EXAMPLES ON CHAPTER IX

1. Find the centre of gravity of a semi-circular arc.

2. Find the position of the centre of inertia of an arch of the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

3. Find the centroid of the area enclosed between $y=x^n$, the x-axis, and the lines $x=a$ and $x=b$, where a and b are positive.

4. Show that the area included between the curve whose equation is $x^2y=x^3+a^3$, the axis of x, and the ordinates at $x=a$ and $x=2a$ is $2a^2$.

Find the coordinates of the centre of mass of this area.

5. Show that the centre of gravity of the quadrant between OX and OY of the ellipse $x^2/a^2+y^2/b^2=1$ is

$(4a/3\pi, 4b/3\pi)$. [*Utkal*, 1956]

6. Find the centroid of a hemispherical surface. [*Patna*, 1941]

7. Find the total mass and the coordinates of the centroid of a quadrant of a circular disc of radius a, the

surface density being proportional to the distance from one of the bounding radii, and having unity for its greatest value.

8. A quadrant of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

revolves about the major axis. Find the centre of gravity of the solid thus generated. [*Travancore*, 1941]

9. Find the depth of the centre of pressure on a submerged rectangular vertical door, of breadth b and height h, the upper edge of the door being parallel to the free surface and at a depth d.

10. Find the moment of inertia of an equilateral triangle about its base.

11. ABC is a uniform equilateral triangular plate of mass M and side a. Find its moments of inertia about each of the bisectors of the angle A. [*Lucknow*, 1940]

12. For the area included between the curves $y^2=4ax$, $x^2=4ay$, find (i) the coordinates of the centroid, (ii) the moment of inertia about the x-axis, assuming in each case a uniform density.

13. Show that the moment of inertia about the x-axis of the portion of the parabola $y^2=4ax$ bounded by the axis and the latus rectum, supposing the surface density at each point to vary as the cube of the abscissa, is $\frac{12}{11}Ma^2$, where M is the mass of the portion. [*Andhra*, 1942]

14. Find the moment of inertia of a hollow circular cylinder about its axis, the external and internal radii being R and r respectively.

15. Find the moment of inertia of a uniform solid sphere of radius a feet, mass m lbs., about any tangent line.

16. Prove that the moment of inertia about an axis through the centre, perpendicular to the plane, of a circular ring whose outer and inner radii are a and b is $\frac{1}{2}m(a^2+b^2)$, where m denotes the mass of the ring.

17. Find the moment of inertia of a circular disc about an axis through its centre perpendicular to its plane, assuming that the density at any point varies as the square of its distance from the centre.

18. Show that the moment of inertia of a right cone with respect to an axis drawn through its vertex perpendicular to its axis is $\frac{3}{5}M(h^2+\frac{1}{4}b^2)$, where h denotes the altitude of the cone, and b the radius of its base.

19. Assuming that the gravitational attraction of the Earth on a particle of mass m at a distance r from its centre varies as m/r^2, show that the work done when the particle falls to the surface from a height h is $mgha/(a+h)$, where a is the radius of the Earth and g the acceleration due to gravity at the surface.

20. A recoil buffer is so adjusted that when the gun has recoiled a distance of x inches, the force resisting the recoil is $W(1-x^2/a^2)$ tons, where W and a are constants. Find the work done when the gun recoils through a distance of b inches.

21. Two cubic feet of gas at a pressure of 100 lb. per square inch expand to a volume of 3 cubic feet. Find the work done if the law of expansion is $pv^n=c$. If $n=1\cdot5$, calculate the work done.

DIFFERENTIAL EQUATIONS

CHAPTER X

EQUATIONS OF THE FIRST ORDER AND THE FIRST DEGREE

10·1. Introduction. Any relation between known functions and an unknown function is called a *differential equation* if it involves the differential coefficient (or coefficients) of the unknown function.

It is usual to denote the unknown function by y.

Finding the unknown function is called *solving* or *integrating* the differential equation. The *solution* or *integral* of the differential equation is also called its primitive, because the differential equation can be regarded as a relation derived from it.

It is not necessary that the solution be an explicit function of the independent variable x. Any relation between x and y, not containing the differential coefficient of y, is called a solution provided y and the differential coefficients of y derived from it satisfy the differential equation. The solution always contains one or more arbitrary constants.

The integral of a function $f(x)$ may be regarded as the solution of the differential equation

$$\frac{dy}{dx}=f(x).$$

We have seen that the most general solution is $y=\int f(x)\,dx+C$, and contains one arbitrary constant.

The unknown function may also be a function of two or more independent variables. In this case the differential equation will involve partial differential coefficients of the

unknown function. Such a differential equation is called a *partial differential equation.* On the other hand, the differential equation which does not involve partial differential coefficients is called an *ordinary differential equation.* Only ordinary differential equations will be considered in this book.

Differential equations are of great importance in applied mathematics, physics and other branches of knowledge, and arise because often we know from physical considerations some relation which involves one or more differential coefficients of the unknown function.

For example, suppose a particle of mass m is falling under gravity, from a great distance, towards the earth. Let its distance from the centre of the earth at time t be x. The attraction there, as we know from the law of gravitation, is proportional to $1/x^2$. Let it be equal to k/x^2. Also the acceleration is represented by d^2x/dt^2. Therefore, by the laws of dynamics

$$m\frac{d^2x}{dt^2}=-\frac{k}{x^2}. \qquad . \quad . \quad . \quad (1)$$

Now we want to express x in terms of t in order to know how the particle moves. The only relation which we know is (1), which is a differential equation. To determine x as a known function of t, we must solve this differential equation.

The *order* of a differential equation is the order of the highest differential coefficient which occurs in it.

The *degree* of a differential equation is the degree of the highest differential coefficient which occurs in it, after the differential equation has been cleared of radicals and fractions.

Thus the differential equation

$$f(x,y)\left(\frac{d^my}{dx^m}\right)^p+\phi(x,y)\left(\frac{d^{m-1}y}{dx^{m-1}}\right)^q+\ldots=0$$

is of order m and degree p.

10·11. Arbitrary constants. In order to find how many arbitrary constants will occur in the solution of a differential equation, let us study how the differential equation can be formed if the primitive (i.e., the solution) is known.

Let the primitive be $f(x, y, a)=0$, where a is an arbitrary constant.

Differentiation gives us a relation between x, y, a and dy/dx, say

$$\phi\left(x, y, \frac{dy}{dx}, a\right)=0.$$

Elimination of a between this and the primitive will give us a relation between x, y and dy/dx, say

$$\psi\left(x, y, \frac{dy}{dx}\right)=0,$$

which is a differential equation of the first order. Hence, looking back, we may expect the solution of a differential equation of the first order to contain one arbitrary constant.

Again, suppose the primitive is

$$f(x, y, a, b)=0,$$

so that there are now two arbitrary constants. We must now have two more relations in order to be able to eliminate a and b. Differentiating successively, we get, say,

$$\phi_1\left(x, y, \frac{dy}{dx}, a, b\right)=0,$$

and $$\phi_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, a, b\right)=0.$$

Elimination of a and b between these three equations will give a relation between x, y, dy/dx, and d^2y/dx^2, say

$$\psi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0,$$

which is a differential equation of the second order. Thus we may expect the solution of a differential equation of the second order to contain two arbitrary constants; and so on.

10·12. General and particular solutions. The solution of a differential equation which contains a number of arbitrary constants equal to the order of the differential equation is called the *general solution* (or complete integral or complete primitive). A solution obtainable from the general solution by giving particular values to the constants is called a *particular solution.*

In counting the arbitrary constants in the general solution, care must be taken to see that they are independent, and not equivalent to a lesser number of arbitrary constants. Thus, although the solution

$$y=A \sin x+B \cos (x+C)$$

appears to contain three arbitrary constants, they are really equivalent to two only.

For,

$$\begin{aligned} A \sin x+B\cos (x+C) &=(A-B\sin C) \sin x+B\cos C\cos x \\ &=\alpha \sin x+\beta \cos x, \text{ say,} \end{aligned}$$

and by giving to α and β suitable values we can evidently reproduce any particular solution which can be obtained by giving particular values to A, B and C. Hence the three constants A, B, C are really equivalent to two only.

Moreover, the general solution can have more than one form, but the arbitrary constants in one form will be related to the arbitrary constants in the other. Thus

$$y=A\cos(x+B)$$

and

$$y=\alpha\sin x+\beta\cos x$$

are both solutions of the differential equation

$$\frac{d^2y}{dx^2}+y=0,$$

as can be easily verified. Each is a general solution containing two arbitrary constants. Expanding the first and comparing with the second, we see that

$$\alpha=-A\sin B,\quad \beta=A\cos B,$$

and conversely $A=\sqrt{(\alpha^2+\beta^2)}$, $B=-\tan^{-1}(\alpha/\beta)$, showing that the constants in one form are related to the constants in the other.

Sometimes that solution of a differential equation is wanted which satisfies some given relation or relations. In such a case some or all of the arbitrary constants will have definite values, depending upon the number of conditions to be satisfied.

Occasionally the solution of a differential equation involves expressions of the form $\int f(x)\,dx$, or $\int f(y)\,dy$, which cannot be evaluated in terms of the known functions. In such cases the differential equation is regarded as solved when the solution has been expressed in terms of integrals of the above-mentioned forms. Of course, integrals of the form $\int f(x)\,dy$ or $\int f(y)\,dx$ or $\int f(x,y)\,dx$, or $\int f(x,y)\,dy$ must not occur in the solution, for the relation between x and y is not known.

Although it is usual to omit the constant of integration in ordinary integration, such constants should never be omitted when solving differential equations. The reason is that the arbitrary constant in the solution of a differential equation is not always merely additive.

10·13. Meaning of dx and dy. In order to avoid fractions, it is usual to write

$$\frac{dy}{dx}=\frac{f_1(x,y)}{f_2(x,y)},$$

where $\frac{dy}{dx}$ is the differential coefficient of y with respect to x, in the form

$$f_1(x,y)\,dx - f_2(x,y)\,dy = 0,$$

which can be obtained from the previous form by regarding the differential coefficient as the quotient of dy by dx. It is not necessary, however, to attach any meaning to the dx and the dy taken separately, as every equation we shall have to deal with can be converted at once to the form in which dy/dx, the differential coefficient of y with respect to x, alone occurs.

10·2. Equation of the first order and first degree. Not all differential equations can be solved. Even equations of the first order and the first degree cannot be solved in every case; they can be solved, however, if they belong to one or the other of the standard forms discussed in the following articles.

10·3. Equations in which the variables are separable. If it is possible to write a differential equation, by the transposition of its term, in the form

$$f_1(x)\,dx = f_2(y)\,dy,$$

we say that the *variables are separable*. Such equations can be solved immediately by integration. For, the above equation is equivalent to

$$f_1(x) = f_2(y)\,\frac{dy}{dx}.$$

Integrating both sides with respect to x, we get

$$\int f_1(x)\,dx = \int f_2(y)\,\frac{dy}{dx}\,dx + c = \int f_2(y)\,dy + c,$$

where c is an arbitrary constant.

Ex. Solve $(x^2 - yx^2)\,dy + (y^2 + xy^2)\,dx = 0.$

Here $\qquad x^2(1-y)\,dy + y^2(1+x)\,dx = 0,$

or $$\frac{1-y}{y^2}\,dy + \frac{1+x}{x^2}\,dx = 0.$$

Integrating, $$-\frac{1}{y} - \log y - \frac{1}{x} + \log x = c,$$

or $$\log(x/y) - (y+x)/xy = c.$$

EXAMPLES

Solve :

1. $(1+x)y\,dx + (1-y)x\,dy = 0.$
2. $(1-x^2)(1-y)\,dx = xy(1+y)\,dy.$ [*Banaras, Geophysics*, '57]
3. $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}.$
4. $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right).$ [*Baroda*, 1959]
5. $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0.$ [*Bombay*, 1959]
6. $\sqrt{(a+x)}\,dy/dx + x = 0.$
7. $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}.$ [*Delhi*, 1947]
8. $dy/dx = e^{x-y} + x^2 e^{-y}.$ [*Sagar*, 1954]

10·4. Homogeneous equations. A differential equation of the form

$$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}, \qquad \ldots \quad (1)$$

where f_1 and f_2 are homogeneous functions of x and

y of the same degree, is called a *homogeneous differential equation.*

Such equations can be solved by taking a new dependent variable v connected with the old one y by the equation

$$y = vx. \qquad \ldots \quad (2)$$

For, on dividing the numerator and the denominator of the expression on the right-hand side of (1) by x^n, where n is the degree of f_1 and f_2, the differential equation (1) takes the form

$$dy/dx = f(y/x).$$

The substitution (2) will, therefore, transform it into an equation of the form

$$v + x\frac{dv}{dx} = f(v).$$

The variables are now separable, and the solution is

$$\int \frac{dx}{x} = \int \frac{dv}{f(v) - v} + c.$$

Replacing v by y/x after integration, we have the final solution.

NOTE. Before solving a homogeneous equation by putting $y = vx$, it is advisable to try if the variables are separable (§ 10·3), or the equation is exact (§ 10·6). For in these cases the differential equation can be solved directly without any substitution.

Ex. Solve $\quad x(x-y)\,dy + y^2\,dx = 0.$

Putting $y = vx$, we have

$$v + x\frac{dv}{dx} = \frac{y^2}{x(y-x)} = \frac{v^2}{v-1}.$$

Therefore $\quad x\dfrac{dv}{dx} = \dfrac{v^2}{v-1} - v = \dfrac{v}{v-1}.$

Separating the variables and integrating,

$$\log x = \int \frac{(v-1)\,dv}{v} + c_1 = v - \log v + c_1,$$

where c_1 is an arbitrary constant,

$$= \log e^v - \log v + \log c,$$

where c is an arbitrary constant.

Hence $$x = \frac{ce^v}{v} = \frac{cxe^{y/x}}{y},$$

i.e., $$y = ce^{y/x}.$$

NOTE. In the above we took a new arbitrary constant c instead of retaining the old one c_1 in order to write the result in an elegant form. Such changes are freely resorted to in the solution of differential equations.

EXAMPLES

Solve the following differential equations :

1. $x + y\dfrac{dy}{dx} = 2y.$
2. $y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx}.$ [*Banaras, Geophysics*, 1956]
3. $x\,dy - y\,dx = \sqrt{(x^2+y^2)}\,dx.$ [*Ujjain*, 1960]
4. $x^2\dfrac{dy}{dx} = \dfrac{y(x+y)}{2}.$ [*Banaras, Geophysics*, 1961]
5. $\dfrac{dy}{dx} = \dfrac{x^2+xy}{x^2+y^2}.$ [*Allahabad*, 1956]
6. $(x^2-y^2)\,dx + 2xy\,dy = 0.$ [*Sagar*, 1954]
7. $\dfrac{dy}{dx} + \dfrac{x^2+3y^2}{3x^2+y^2} = 0.$ [*Agra*, 1951]
8. $x(x-y)\,dy/dx = y(x+y).$ [*Lucknow*, 1951]
9. $(x^2+2xy)\,dy/dx + 2xy + y^2 + 3x^2 = 0.$
10. $(x^3-3xy^2)\,dx = (y^3-3x^2y)\,dy.$ [*Aligarh*, 1958]
11. $x^2y\,dx - (x^3+y^3)\,dy = 0.$ [*Ujjain*, 1960]
12. $\{x\cos(y/x) + y\sin(y/x)\}y$
    $-\{y\sin(y/x) - x\cos(y/x)\}x\,dy/dx = 0.$ [*Raj.*, '54]

10·41. Equations reducible to a homogeneous form. Equations of the form

$$\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C}$$

can be reduced to a homogeneous form by taking new variables ξ and η, related to x and y by the equations

$$x=\xi+h, \quad y=\eta+k,$$

where h and k are constants which are yet to be chosen. With these substitutions,

$$\frac{dy}{dx}=\frac{d}{dx}(\eta+k)=\frac{d\eta}{dx}=\frac{d\eta}{d\xi}\cdot\frac{d\xi}{dx}=\frac{d\eta}{d\xi}.$$

Hence the differential equation assumes the form

$$\frac{d\eta}{d\xi}=\frac{a\xi+b\eta+(ah+bk+c)}{A\xi+B\eta+(Ah+Bk+C)}.$$

Now choose h and k so that

$$\left.\begin{aligned} ah+bk+c=0,\\ Ah+Bk+C=0.\end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

Then the differential equation becomes homogeneous and can be solved by the substitution $\eta=v\xi$. Replacing ξ and η in the solution thus obtained by $x-h$ and $y-k$ respectively, we shall get the solution in terms of the original variables.

A SPECIAL CASE.

The solution of equations (1) gives

$$\frac{h}{bC-Bc}=\frac{k}{cA-Ca}=\frac{1}{aB-Ab},$$

and so fails if $aB-Ab=0$, i.e., if $\dfrac{a}{A}=\dfrac{b}{B}$.

In this case the differential equation is of the form

$$\frac{dy}{dx}=\frac{ax+by+c}{max+mby+C},$$

and can be recognised by a preliminary examination. Such a differential equation can be solved by putting v for $ax+by$ and getting rid of y. The transformed differential equation is

$$\frac{dv}{dx}=a+b\frac{v+c}{mv+C},$$

in which the variables are separable.

NOTE. Before applying the method of this article, it is worthwhile to see if the equation

$$(ax+by+c)\,dx-(Ax+By+C)\,dy=0$$

is exact (§ 10·6). It will be so if $b=-A$. In this case the differential equation can be solved directly without any substitution.

Ex. 1. Solve $(x-y-2)\,dx+(x-2y-3)\,dy=0$. [*Gor.*, '60]

Putting $x=\xi+h$, $y=\eta+k$, we get

$$\frac{d\eta}{d\xi}=-\frac{\xi-\eta+h-k-2}{\xi-2\eta+h-2k-3}. \quad \ldots \quad (1)$$

Choose h and k so that

$$h-k-2=0,$$

and

$$h-2k-3=0.$$

Solving these, we get $h=1$, $k=-1$.

With these values of h and k, (1) becomes

$$\frac{d\eta}{d\xi}=\frac{\eta-\xi}{\xi-2\eta}.$$

Putting $\eta=v\xi$, we have

$$v+\xi\frac{dv}{d\xi}=\frac{1-v}{2v-1},$$

or

$$\xi\frac{dv}{d\xi}=\frac{1-2v^2}{2v-1}.$$

Therefore

$$\log \xi + c_1 = -\int \frac{2v-1}{2v^2-1}\,dv = -\tfrac{1}{2}\int \frac{4v\,dv}{2v^2-1} + \tfrac{1}{2}\int \frac{dv}{v^2-\tfrac{1}{2}}$$

$$= -\tfrac{1}{2}\log(2v^2-1) + \frac{\sqrt{2}}{4}\log\frac{v-1/\sqrt{2}}{v+1/\sqrt{2}}.$$

On writing $x-1$ for ξ and $(y+1)/(x-1)$ for v, we shall get the solution in terms of x and y.

Ex. 2. Solve $(x-y-2)\,dx-(2x-2y-3)\,dy=0$. [*Bar.*, '59]

We notice that the coefficients of x and y in the numerator and denominator of the expression for dy/dx are proportional. We, therefore, proceed as follows :

Put $x-y=v$.

Then $\dfrac{dv}{dx}=1-\dfrac{dy}{dx}=1-\dfrac{x-y-2}{2x-2y-3}=1-\dfrac{v-2}{2v-3}=\dfrac{v-1}{2v-3}$.

Therefore $\quad x+c=\displaystyle\int \frac{2v-3}{v-1}\,dv=\int\left\{2-\frac{1}{v-1}\right\}dv$

$$=2v-\log(v-1)=2(x-y)-\log(x-y-1),$$

or $\quad \log(x-y-1)=x-2y-c.$

EXAMPLES

Integrate the following differential equations :

1. $(x-y)\,dy=(x+y+1)\,dx$. [*Rajasthan*, 1960]
2. $\dfrac{dy}{dx}=\dfrac{x+2y-3}{2x+y-3}$. [*Aligarh*, 1960]
3. $\dfrac{dy}{dx}=\dfrac{y-x+1}{y+x+5}$. [*Aligarh*, 1951]
4. $\dfrac{dy}{dx}=\dfrac{x+y+1}{2x+2y+3}$. [*Agra*, 1954]
5. $(2x+4y+3)y'=2y+x+1$. [*Rajasthan*, 1960]
6. $\dfrac{4x+6y+5}{3y+2x+4}\cdot\dfrac{dy}{dx}=1$. [*Agra*, 1958]
7. $(6x+2y-10)(dy/dx)-2x-9y+20=0$. [*Gorakh.*, '59]
8. $(2x+3y-5)(dy/dx)+3x+2y-5=0$. [*Agra*, 1956]

10·5. Linear equations. A differential equation is said to be *linear* when the dependent variable y and its differential coefficients occur only in the first degree. The coefficients of y and of its differential coefficients may be any functions of x.

The linear differential equation of the first order is, therefore, of the form

$$\frac{dy}{dx}+Py=Q,$$

where P and Q are any functions of x.

To solve such an equation, multiply both sides by

$$e^{\int P\,dx}.$$

The left-hand side now is evidently the differential coefficient of

$$ye^{\int P\,dx};$$

so that the solution of the differential equation is

$$ye^{\int P dx}=\int Qe^{\int P\,dx}dx+c.$$

NOTE 1. The factor $e^{\int P\,dx}$, by multiplying by which the left-hand side of the differential equation (written as above) becomes a differential coefficient of some function of x and y, is called an *integrating factor* of the differential equation.

2. Sometimes a given differential equation becomes linear if we *take y as the independent variable* and x as the dependent variable. Thus, by this device

$$(x+y+a)\frac{dy}{dx}=y^2+b$$

can be written as

$$(y^2+b)\frac{dx}{dy}-x=y+a,$$

which is a linear differential equation.

Ex. Solve $x(1-x^2)\,dy+(2x^2y-y-ax^3)\,dx=0.$

The given equation is equivalent to

$$\frac{dy}{dx}+\frac{2x^2-1}{x(1-x^2)}\,y=\frac{ax^2}{1-x^2},$$

which is linear.

Now
$$\int\frac{2x^2-1}{x(1-x^2)}\,dx=\int\frac{1-2x^2}{x(x-1)(x+1)}\,dx$$
$$=-\int\left\{\frac{1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}\right\}dx$$
$$=\log\{1/x\sqrt{(x^2-1)}\}.$$

Hence the integrating factor is $1/x\sqrt{(x^2-1)}$, and the solution is

$$\frac{y}{x\sqrt{(x^2-1)}}=c+a\int\frac{x^2}{1-x^2}\cdot\frac{1}{x\sqrt{(x^2-1)}}\,dx$$
$$=c-\tfrac{1}{2}a\int\frac{dt}{t^{3/2}},\text{ where } t=x^2-1,$$
$$=c+a(x^2-1)^{-1/2},$$

i.e.,
$$y=cx\sqrt{(x^2-1)}+ax.$$

Examples

Solve

1. $x\dfrac{dy}{dx}+y=x^2+3x+2.$ [*Aligarh*, 1956]

2. $(x+a)\dfrac{dy}{dx}-3y=(x+a)^5.$ 3. $\dfrac{dy}{dx}+ay=e^{mx}.$ [*Baroda*,'56]

4. $y'+y=x^{-2}.$ 5. $\dfrac{dy}{dx}=mx+ny+q.$

6. $\dfrac{dy}{dx}+\dfrac{x}{1+x^2}\,y=\dfrac{1}{2x(1+x^2)}.$

7. $(1-x^2)\dfrac{dy}{dx}+2xy=x(1-x^2)^{1/2}.$ [*Madras*, 1955]

8. $(x^2+1)\dfrac{dy}{dx}+2xy=4x^2.$ [*Banaras*, 1955]

9. $x(x-1)\dfrac{dy}{dx}-(x-2)y=x^3(2x-1)$. [*Lucknow*, 1962]

10. $(1+y+x^2y)\,dx+(x+x^3)\,dy=0$.

11. $\dfrac{dy}{dx}+\dfrac{2}{x}y=\sin x$. [*Jammu*, 1956]

12. $\sec x\dfrac{dy}{dx}=y+\sin x$. [*Rajasthan*, 1962]

13. $\dfrac{dy}{dx}=y\tan x-2\sin x$. [*Panjab*, 1957]

14. $(1+x^2)\dfrac{dy}{dx}+2xy=\cos x$. [*Ban. Geoph.*, '61]

15. $\sin x\dfrac{dy}{dx}+3y=\cos x$.

16. $\sin 2x\dfrac{dy}{dx}=y+\tan x$.

17. $\dfrac{dy}{dx}+y\tan x-\sec x=0$. [*Jammu*, 1950]

18. $x\dfrac{dy}{dx}-y=2x^2\operatorname{cosec} 2x$. [*Allahabad*, 1955]

19. $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy$. [*Panjab*, 1962]

20. $(y-x)\dfrac{dy}{dx}=a^2$. [*Travancore*, 1957]

21. $(x+2y^3)\dfrac{dy}{dx}=y$. [*P.S.C., U.P.*, 1953]

22. $(2x-10y^3)\dfrac{dy}{dx}+y=0$. [*P.S.C., U.P.*, 1955]

23. $Dy+y(1-x^2)^{-3/2}=\{x+\sqrt{(1-x^2)}\}(1-x^2)^{-2}$,
where $D\equiv d/dx$.

24. Integrate $(1+x^2)\dfrac{dy}{dx}+2yx-4x^2=0$, and obtain the cubic curve satisfying this equation and passing through the origin.

25. Solve $\dfrac{dy}{dx}+\dfrac{y}{x}=x^2$, given $y=1$ when $x=1$.

10·51. Equations reducible to the linear form. Equations of the form

$$\frac{dy}{dx}+Py=Qy^n,$$

where P and Q are functions of x alone, can be reduced to the linear form by dividing by y^n and putting y^{-n+1} equal to v. For, on dividing by y^n, we get

$$y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q,$$

and on substituting v for y^{-n+1} we have

$$\frac{1}{(-n+1)}\frac{dv}{dx}+Pv=Q,$$

which is a linear differential equation in v.

This equation is often called Bernoulli's equation*.

Ex. Solve $\qquad x\,Dy+y=xy^3.$

Dividing by y^3, we have $xy^{-3}\,Dy+y^{-2}=x$.

Putting $y^{-2}=v$, and therefore $-2y^{-3}Dy=Dv$, the differential equation becomes

$$-\tfrac{1}{2}x\,Dv+v=x,$$

or
$$\frac{dv}{dx}-\frac{2}{x}v=-2.$$

The integrating factor is $e^{-2\log x}$, i.e., $1/x^2$. So the solution is

$$\frac{v}{x^2}=-2\int\frac{dx}{x^2}+c=\frac{2}{x}+c,$$

i.e.,
$$(2+cx)xy^2=1.$$

*Named after James Bernoulli (1654-1705), Professor of Mathematics in the University of Basel (Switzerland), who first studied it. He was a staunch friend of Leibnitz, and was one of the first to apply the differential calculus successfully to a great variety of problems.

EXAMPLES

Solve

1. $\frac{dy}{dx}+\frac{y}{x}=y^2.$ [*Gauhati*, 1955]

2. $2\frac{dy}{dx}=\frac{y}{x}+\frac{y^2}{x^2}.$ [*Nagpur*, 1953]

3. $(1-x^2)\frac{dy}{dx}+xy=xy^2.$ [*Jammu*, 1953]

4. $x\frac{dy}{dx}+y=y^2\log x.$ [*Allahabad*, 1960]

5. $\frac{dy}{dx}+xy=y^2 e^{x^2/2}\sin x.$

6. $\frac{dy}{dx}+\frac{y}{x}=y^2\sin x.$

7. $(1+x^2)y'=xy-y^2.$

8. $3\frac{dy}{dx}+\frac{2}{x+1}y=\frac{x^3}{y^2}.$ [*Delhi*, 1960]

9. $\frac{dy}{dx}=x^3y^3-xy.$ [*Gorakhpur*, 1960]

10. $xy-\frac{dy}{dx}=y^3e^{-x^2}.$ [*Karnatak*, 1957]

11. $2\frac{dy}{dx}-y\sec x=y^3\tan x.$

12. $x^2y-x^3\frac{dy}{dx}=y^4\cos x.$

13. $x\frac{dy}{dx}+y=x^3y^6.$ [*Banaras*, 1953]

14. $y(2xy+e^x)\,dx-e^x\,dy=0.$ [*Lucknow*, 1951]

15. $\cos x\,dy=y(\sin x-y)\,dx.$ [*Aligarh*, 1950]

10·6. Exact differential equations. A differential equation is said to be exact if it can be derived from its primitive directly by differentiation, without any subsequent multiplication, elimination, etc. Thus the differential equation

$$M+N\frac{dy}{dx}=0, \quad \dots \quad (1)$$

where M and N are functions of x and y, is exact if it can be obtained directly by differentiating an equation of the type $u=c$, where u is some function of x and y, and c is an arbitrary constant.

Now $$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}.$$

Hence the equation (1) must be the same as

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=0.$$

Therefore, *a necessary condition that the equation* $M\,dx+N\,dy=0$ *be exact is that* $M=\partial u/\partial x$, $N=\partial u/\partial y$, or, eliminating u, that

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}}=\frac{\partial \mathbf{N}}{\partial \mathbf{x}}.$$

This condition is also sufficient; i.e., if $\partial M/\partial y=\partial N/\partial x$, then $M\,dx+N\,dy=0$ must be an exact differential equation. For, if we put $\int M\,dx=U$, then

$$\frac{\partial U}{\partial x}=M, \text{ so that } \frac{\partial^2 U}{\partial y\,\partial x}=\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}, \text{ by hypothesis,}$$

i.e., $$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right).$$

It follows that $N=\dfrac{\partial U}{\partial y}+f(y)$, where $f(y)$ is a function of y alone.

Therefore $$M+N\frac{dy}{dx}=\frac{\partial U}{\partial x}+\left\{\frac{\partial U}{\partial y}+f(y)\right\}\frac{dy}{dx} \qquad (2)$$

$$=\frac{d}{dx}\left\{U+\int f(y)\,\frac{dy}{dx}\,dx\right\}=\frac{d}{dx}\{U+F(y)\},$$

showing that $M+N(dy/dx)=0$ is an exact equation.

If we find that an equation $M\,dx+N\,dy=0$ satisfies the condition

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

and so is exact, we can integrate it as follows:

First integrate M with respect to x as if y were a constant. Then integrate N with respect to y, retaining only those terms which have not been already obtained by the integration of M. The sum of the expressions thus obtained equated to an arbitrary constant will be the solution.

The reason for this procedure becomes obvious when we examine equation (2).

Ex. 1. Solve the equation

$$(1+4xy+2y^2)\,dx+(1+4xy+2x^2)\,dy=0.$$

Here $$\frac{\partial M}{\partial y}=4x+4y,\quad \frac{\partial N}{\partial x}=4y+4x.$$

These being equal, it follows that the differential equation is exact. Integrating $1+4xy+2y^2$ with respect to x, while regarding y as a constant, we get $x+2x^2y+2xy^2$.

Again, the only new term obtained on integrating $1+4xy+2x^2$ with respect to y is y.

Hence the solution of the given differential equation is

$$x+2x^2y+2xy^2+y=c.$$

Ex. 2. Solve $y\sin 2x\,dx-(1+y^2+\cos^2 x)\,dy=0$.

It is easy to verify that the equation is exact. Integrating M with respect to x we get

$$-\tfrac{1}{2}y\cos 2x. \quad . \quad . \quad . \quad (1)$$

Integrating N with respect to y, we get

$$-y-\tfrac{1}{3}y^3-y\cos^2 x,\text{ i.e., }-y-\tfrac{1}{3}y^3-\tfrac{1}{2}(1+\cos 2x)y.$$

Omitting from this the term already occurring in (1), and adding the rest of the terms to (1), we see that the solution is

$$-\tfrac{1}{2}\,y\cos 2x-\tfrac{3}{2}y-\tfrac{1}{3}y^3=c_1,$$

or $$y\cos 2x+3y+\tfrac{2}{3}y^3=c.$$

[We can easily verify by differentiation that this solution is correct.]

Examples

Show that the following equations are exact, and solve them.

1. $(2ax+by)\,y\,dx+(ax+2by)\,x\,dy=0.$
2. $x\,dx+y\,dy=\dfrac{a^2(x\,dy-y\,dx)}{x^2+y^2}.$ [*Banaras*, 1960]
3. $(x^2-ay)\,dx=(ax-y^2)\,dy.$ [*Ban. Geoph.*, 1961]
4. $(e^y+1)\cos x\,dx+e^y\sin x\,dy=0.$ [*Allahabad*, 1960]
5. $\cos x(\cos x-\sin\alpha\sin y)\,dx$
   $+\cos y(\cos y-\sin\alpha\sin x)\,dy=0.$ [*Nagpur*, 1952]
6. $\{y(1+1/x)+\cos y\}\,dx+\{x+\log x-x\sin y\}\,dy=0.$ [*Ujjain*, 1960]
7. $(1+e^{x/y})\,dx+e^{x/y}(1-x/y)\,dy=0.$ [*Gujarat*, 1959]
8. Solve $\dfrac{dy}{dx}+\dfrac{ax+hy+g}{hx+by+f}=0.$ [*Delhi*, 1957]

10·7. Integrating factors. Equations which are not exact can often be made exact by multiplying them by some function of x and y. Such a function is called an *integrating factor*.

The number of integrating factors for an equation
$$M\,dx+N\,dy=0$$
is infinite. For, if μ is an integrating factor, then by definition,
$$\mu(M+N\,dy/dx)$$
must be the differential coefficient of some function u of x and y.

It follows that $f(u)\,.\,\mu$, where $f(u)$ is any function of u, is also an integrating factor; for multiplication by it transforms $M+N\,dy/dx$ into
$$f(u)\,.\,\mu\left(M+N\frac{dy}{dx}\right),\quad \text{i.e.,}\ f(u)\frac{du}{dx},$$

which is the differential coefficient, with respect to x, of $F(u)$, where $\int f(u)\,du=F(u)$.

This proposition, however, does not help us in finding integrating factors.

(i) *Integrating factor found by inspection.* Sometimes an integrating factor can be found by inspection.

Ex. Integrate $(x^3e^x-my^2)\,dx+mxy\,dy=0$.

We know that x^ne^x when differentiated gives two terms, each containing e^x as a factor. Now the given differential equation has only one term which involves e^x. Hence it must have come from the differentiation of e^x alone. To make the equation exact, we shall, therefore, try division by x^3. We get thus

$$e^x-m\frac{y^2-xyy'}{x^3}=0.$$

The last term resembles the expression obtained by differentiating a quotient, but requires a little re-arrangement to make it exactly a differential coefficient. We write the equation as

$$e^x+\tfrac{1}{2}m\frac{-2xy^2+2x^2yy'}{x^4}=0,$$

and see that the solution is

$$e^x+\tfrac{1}{2}\,my^2/x^2=c.$$

(ii) *If the equation* $M\,dx+N\,dy=0$ *has the form*

$$f_1(xy)\,y\,dx+f_2(xy)\,x\,dy=0,$$

and $Mx-Ny\neq 0$, *an integrating factor is*

$$1/(Mx-Ny).$$

For $M+N\,dy/dx$ can be written as

$$\tfrac{1}{2}\left\{(Mx+Ny)\left(\frac{1}{x}+\frac{1}{y}\cdot\frac{dy}{dx}\right)+(Mx-Ny)\left(\frac{1}{x}-\frac{1}{y}\cdot\frac{dy}{dx}\right)\right\},$$

i.e., as $\tfrac{1}{2}\left\{(Mx+Ny)\dfrac{d}{dx}\log(xy)+(Mx-Ny)\dfrac{d}{dx}\log\dfrac{x}{y}\right\}$.

Multiplication by $1/(Mx-Ny)$ gives

$$\tfrac{1}{2}\left(\frac{Mx+Ny}{Mx-Ny}\right)\frac{d}{dx}\log(xy)+\tfrac{1}{2}\frac{d}{dx}\log\frac{x}{y}.$$

Now
$$\frac{Mx+Ny}{Mx-Ny}=\frac{f_1(xy)\,xy+f_2(xy)\,xy}{f_1(xy)\,xy-f_2(xy)\,xy}$$
$$=F(xy)=\phi\{\log(xy)\}.$$

So the multiplication by $1/(Mx-Ny)$ reduces the given differential equation to the form

$$\phi(\log xy)\frac{d}{dx}(\log xy)+\frac{d}{dx}\log\frac{x}{y}=0,$$

which is evidently an exact differential equation.

Ex. Solve $(x^2y^2+xy+1)\,y\,dx+(x^2y^2-xy+1)\,x\,dy=0.$

The integrating factor is
$1/\{(x^2y^2+xy+1)xy-(x^2y^2-xy+1)xy\}$, i.e., $1/2x^2y^2$.

Multiplication by this transforms the given differential equation into

$$\tfrac{1}{2}\left(1+\frac{1}{xy}+\frac{1}{x^2y^2}\right)y\,dx+\tfrac{1}{2}\left(1-\frac{1}{xy}+\frac{1}{x^2y^2}\right)x\,dy=0.$$

As the equation is now exact, we can apply to it the method of § 10·6. We get as the solution

$$\tfrac{1}{2}(xy+\log x-1/xy)-\tfrac{1}{2}\log y=c_1,$$

or, if $2c_1=-\log c$,

$$x^2y^2+xy\log(cx/y)=1.$$

NOTE. It has been assumed that $Mx-Ny$ is not equal to zero. If, however, this expression is zero, it follows that $M/y=N/x$, so that the differential equation $M\,dx+N\,dy=0$ reduces in this case by algebraic simplification to

$$y\,dx+x\,dy=0,$$

the solution of which is $xy=c$.

(iii) *If*
$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$
is a function of x alone, say $f(x)$, then
$$e^{\int f(x)\,dx}$$
is an integrating factor of the equation $M\,dx+N\,dy=0$.

For, upon multiplication by this factor the new differential equation satisfies the condition of being exact, as is easily verified.

Ex. Solve the equation

$$(20x^2+8xy+4y^2+3y^3)y\,dx+4(x^2+xy+y^2+y^3)x\,dx=0.$$

Here $\qquad \partial M/\partial y=20x^2+16xy+12y^2+12y^3,$

and $\qquad \partial N/\partial x=12x^2+8xy+4y^2+4y^3.$

Therefore $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=\dfrac{8x^2+8xy+8y^2+8y^3}{4(x^2+xy+y^2+y^3)x}=\dfrac{2}{x}.$

Hence the integrating factor is $e^{2\log x}$, i.e., x^2. Multiplying the differential equation by it, we get

$$(20x^4+8x^3y+4x^2y^2+3x^2y^3)y\,dx+\text{etc.}=0.$$

Since the differential eqn. is now exact, the solution is

$$(4x^5+2x^4y+\tfrac{4}{3}x^3y^2+x^3y^3)y=c.$$

(iv) *If* $\qquad \dfrac{1}{M}\left(\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}\right)$

is a function of y alone, say $f(y)$, then

$$e^{\int f(y)\,dy}$$

is an integrating factor of the equation $M\,dx+N\,dy=0$.

This can be proved to be true exactly in the same way as in rule (iii).

(v) *An integrating factor for an equation of the form*

$$x^a y^b(my\,dx+nx\,dy)+x^r y^s(py\,dx+qx\,dy)=0,$$

a, b, m, n, r, s, p, q being constants, is

$$\mathbf{x}^h\mathbf{y}^k,$$

where h, k can be obtained by applying the condition that after multiplication by $x^h y^k$, the equation must become exact.

Multiplying by the proposed integrating factor, the equation becomes

$$(mx^{a+h}y^{b+k+1}+px^{r+h}y^{s+k+1})\,dx$$
$$+(nx^{a+h+1}y^{b+k}+qx^{r+h+1}y^{s+k})\,dy=0.$$

Since this equation must be an exact one, $\partial M/\partial y$ must be equal to $\partial N/\partial x$, i.e.,

$$m(b+k+1)x^{a+h}y^{b+k}+p(s+k+1)x^{r+h}y^{s+k}$$
$$=n(a+h+1)x^{a+h}y^{b+k}+q(r+h+1)x^{r+h}y^{s+k}.$$

This will be satisfied if

$$m(b+k+1)=n(a+h+1),$$

and

$$p(s+k+1)=q(r+h+1).$$

These two equations determine h and k.

Ex. Solve $\quad (3x+2y^2)y\,dx+2x(2x+3y^2)\,dy=0.$

Upon trial it is found that the equation is not exact. But it can be put in the form

$$x(3y\,dx+4x\,dy)+y^2(2y\,dx+6x\,dy)=0.$$

Hence there must be an integrating factor of the form x^hy^k. Multiplying the original equation by it, we get

$$(3x^{h+1}y^{k+1}+2x^hy^{k+3})\,dx+(4x^{h+2}y^k+6x^{h+1}y^{k+2})\,dy=0.$$

If this is exact, we must have

$$3(k+1)x^{h+1}y^k+2(k+3)x^hy^{k+2}$$
$$=4(h+2)x^{h+1}y^k+6(h+1)x^hy^{k+2}.$$

This is satisfied if $\quad 3(k+1)=4(h+2),$

and $\quad 2(k+3)=6(h+1).$

Solving these, we find $h=1,\ k=3.$

So the integrating factor is xy^3. Upon multiplication by it the original equation becomes

$$(3x^2y^4+2xy^6)\,dx+\text{etc.}=0.$$

Hence the solution is

$$x^3y^4+x^2y^6=c, \quad \text{or} \quad x^2y^4(x+y^2)=c.$$

EXAMPLES

Find the integrating factor of the following differential equations; also solve them.

1. $y\,dx - x\,dy + (1+x^2)\,dx + x^2 \sin y\,dy = 0$. [*Rajasthan*, '59]
2. $y(axy + e^x)\,dx - e^x\,dy = 0$. [*Baroda*, 1959]
3. $(x^4y^4 + x^2y^2 + xy)y\,dx + (x^4y^4 - x^2y^2 + xy)x\,dy = 0$.
4. $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0$. [*Nagpur*, 1953]
5. $(xy \sin xy + \cos xy)y\,dx + (xy \sin xy - \cos xy)x\,dy = 0$. [*Lucknow*, 1959]
6. $(y + \frac{1}{3}y^3 + \frac{1}{2}x^2)\,dx + \frac{1}{4}(1+y^2)x\,dy = 0$. [*Banaras*, 1955]
7. $(7x^4y + 2xy^2 - x^3)\,dx + (x^4 + xy)x\,dy = 0$.
8. $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$. [*P.S.C., U.P.*, 1958]
9. $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$. [*P.S.C.,U.P.*, 1955]
10. $(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$. [*Agra*, 1949]
11. $x(3y\,dx + 2x\,dy) + 8y^4(y\,dx + 3x\,dy) = 0$.
12. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$. [*Baroda*, 1959]

10·8. Change of variables. A suitable substitution often reduces a given differential equation which does not directly come under any of the forms discussed so far to one of these forms. This device, known as the change of the dependent or the independent variable (as the case may be), will be used in the succeeding chapter also.

Ex. Solve $\sec^2 y\,(dy/dx) + 2x \tan y = x^3$. [*Agra*, 1960]

Put $\tan y = v$. Then $(\sec^2 y)y' = v'$. So the differential equation becomes

$$\frac{dv}{dx} + 2xv = x^3,$$

which is linear. The solution is

$$ve^{x^2} = c + \int x^3 e^{x^2}\,dx = c + \tfrac{1}{2}(x^2 - 1)e^{x^2},$$

i.e.,
$$\tan y = ce^{-x^2} + \tfrac{1}{2}(x^2 - 1).$$

EXAMPLES

Solve

1. $(x-y^2)\, dx+2xy\, dy=0.$ [*Allahabad*, 1958]
2. $(x^3+y^2+2)\, dx+2y\, dy=0.$
3. $x\dfrac{dy}{dx}+y\log y=xye^x.$ [*Allahabad*, 1958]
4. $\dfrac{dy}{dx}-\dfrac{\tan y}{1+x}=(1+x)e^x \sec y.$ [*Ujjain*, 1960]
5. $\dfrac{dy}{dx}=\dfrac{e^y}{x^2}-\dfrac{1}{x}.$ [*Gorakhpur*, 1959]
6. $\cos(x+y)\, dy=dx.$ [Hint. Put $x+y=v$.] [*P.S.C., U.P.*, '60]
7. $(x+y)^2\dfrac{dy}{dx}=a^2.$ [*Allahabad*, 1960]
8. $dy/dx=(4x+y+1)^2.$ [*Banaras*, 1960]
9. $\dfrac{dy}{dx}=e^{x-y}(e^x-e^y).$ [*Allahabad*, 1958]
10. $\dfrac{x\, dx+y\, dy}{x\, dy-y\, dx}=\sqrt{\left(\dfrac{a^2-x^2-y^2}{x^2+y^2}\right)}.$ [Hint. Change to polars.] [*Poona*, 1959]

EXAMPLES ON CHAPTER X

Solve*

1. $\dfrac{dy}{dx}+y\cos x+y^3\cos x\sin^2 x=0.$ [*Nagpur*, 1956]
2. $(x+y)(dx-dy)=dx+dy.$ [*Delhi*, 1959]
3. $(x^2+y^2)\, dx-2xy\, dy=0.$ [*Delhi*, 1960]
4. $y+2\dfrac{dy}{dx}=y^3(x-1).$ 5. $x\dfrac{dy}{dx}+\dfrac{y^2}{x}=y.$ [*Agra*, 1954]
6. $x\, Dy-y=x\sqrt{(x^2+y^2)}.$ [*Agra*, 1956]

*These examples are purposely not arranged in any special order.

7. $(x^2+y^2+x)\,dx-(2x^2+2y^2-y)\,dy=0.$ [*Banaras*, 1957]

8. $\dfrac{dy}{dx}+\dfrac{3}{x}y=x^2.$ 9. $\cos^2 x\dfrac{dy}{dx}+y=\tan x.$ [*Osmania*, '60]

10. $\dfrac{dy}{dx}+\dfrac{2x}{1+x^2}y=\dfrac{1}{(1+x^2)^2}$, given that $y=0$ when $x=1$.

11. $(1+xy)y\,dx+(1-xy)x\,dy=0.$ [*Agra*, 1951]

12. $(a^2-2xy-y^2)\,dx-(x+y)^2\,dy=0.$ [*Annam.*, 1950]

13. $(x^2+y^2+a^2)y\dfrac{dy}{dx}+x(x^2+y^2-a^2)=0.$ [*Lucknow*, '55]

14. $\dfrac{dy}{dx}+\dfrac{3x^2y}{1+x^3}=\dfrac{\sin^2 x}{1+x^3}.$ [*Agra*, '56] 15. $x^2\dfrac{dy}{dx}+y=1.$

16. $3x^2y^2+\cos(xy)-xy\sin(xy)+\dfrac{dy}{dx}\left\{2x^3y-x^2\sin(xy)\right\}=0.$

17. $\dfrac{dy}{dx}=2y\tan x+y^2\tan^2 x.$ [*Delhi*, 1949]

18. $(2x^2y-3y^2)\,dx+(2x^3-12xy+\log y)\,dy=0.$

19. $y^2(y\,dx+2x\,dy)-x^2(2y\,dx+x\,dy)=0.$ [*Bombay*, 1950]

20. $(1-x^2)\dfrac{dy}{dx}-xy=x^3y^3.$ [*Sagar*, 1950]

21. $\left(\dfrac{x+y-a}{x+y-b}\right)\dfrac{dy}{dx}=\dfrac{x+y+a}{x+y+b}.$ [*Allahabad*, 1962]

22. $x\cos x\dfrac{dy}{dx}+y(x\sin x+\cos x)=1.$ [*Allahabad*, 1960]

23. $1+y^2+(x-e^{-\tan^{-1}y})\dfrac{dy}{dx}=0.$ [*P.S.C., U.P.*, 1958]

24. $(x-y)^2\dfrac{dy}{dx}=a^2.$ [*Rajasthan*, 1958]

25. $\dfrac{dy}{dx}+\dfrac{y}{(1-x)\sqrt{x}}=1-\sqrt{x}.$ [*Rajasthan*, 1959]

26. $(x^2+2xy-y^2)\,dx+(y^2+2xy-x^2)\,dy=0.$ [*Gujarat*, '61]

27. $x^2y\dfrac{dy}{dx}=xy^2-e^{-1/x^3}$. [*Lucknow*, 1960]

28. $\dfrac{dy}{dx}=e^{3x-2y}+x^2e^{-2y}$.

29. $\dfrac{dy}{dx}=\dfrac{x^2+y^2+1}{2xy}$. [*Poona*, 1954]

30. $xy^3(y\,dx+2x\,dy)+(3y\,dx+5x\,dy)=0$.

31. $\dfrac{dy}{dx}+\dfrac{y}{x}=\sin x$. [*Agra*, 1949]

32. $\dfrac{dy}{dx}=\dfrac{2x-y+1}{x+2y-3}$. [*Agra*, 1959]

33. $dy/dx+y\cos x=y^n\sin 2x$.

34. $(1+x)\dfrac{dy}{dx}-xy=1-x$.

35. $\dfrac{dy}{dx}+x\sin 2y=x^3\cos^2 y$. [*Agra*, 1963]

36. $y(2x^2y+e^x)\,dx-(e^x+y^3)\,dy=0$. [*Vikram*, 1961]

37. $3e^x\tan y+(1-e^x)\sec^2 y\dfrac{dy}{dx}=0$. [*Allahabad*, 1948]

38. $(x+y+1)\dfrac{dy}{dx}=1$. [*Rajasthan*, 1957]

39. $(2x+3y-5)\dfrac{dy}{dx}+2x+3y-1=0$.

40. $x(x-1)\dfrac{dy}{dx}-y=x^2(x-1)^2$. [*Lucknow*, 1948]

41. $x\dfrac{dy}{dx}+2y-x^2\log x=0$.

42. $(2x^2y-3y^4)\,dx+(3x^3+2xy^3)\,dy=0$. [*Lucknow*, 1960]

43. The distance x descended by a person falling by means of a parachute satisfies the differential equation

$$\left(\frac{dx}{dt}\right)^2=k^2(1-e^{-2gx/k^2}),$$

where k and g are constants, and $x=0$ when $t=0$.

Show that

$$x=\frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right).$$

44. If the equation $P\,dx+Q\,dy=0$ can be made exact by means of an integrating factor μ which is a function of x alone, show that

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

should be independent of y.

45. If $2\int v\,dx=v-\log(1+v)+A$, where A is some constant and v is a function of x which is zero when $x=0$, prove that

$$v=2e^x\sinh x.$$ [*Allahabad*, 1959]

46. Solve

$$\frac{dy}{dx}+2y\tan x=\sin x,$$

given that $y=0$ when $x=\frac{1}{3}\pi$. [*Banaras*, 1960]

47. Show that the equation

$$x(x^2+3y^2)\,dx+y(y^2+3x^2)\,dy=0$$

is exact, and solve it. [*Banaras*, 1948]

48. Prove that $1/(x+y+1)^4$ is an integrating factor of

$$(2xy-y^2-y)\,dx+(2xy-x^2-x)\,dy=0,$$

and find the solution of this equation.

49. Show that the equation

$$(4x+3y+1)\,dx+(3x+2y+1)\,dy=0$$

represents a family of hyperbolas having as asymptotes the lines $x+y=0$, $2x+y+1=0$. [*Lucknow*, 1959]

50. Find the integrating factor of

$$y\sec^2 x\,dx+(y+7)\tan x\,dy=0,$$

and solve it. Verify the result by solving the equation by separation of variables.

CHAPTER XI

EQUATIONS OF THE FIRST ORDER, BUT NOT OF THE FIRST DEGREE

11·1. Equations solvable for p. In differential equations which involve dy/dx in a degree higher than one, it is usual to use p to denote dy/dx. Now suppose a differential equation can be solved for p and is of the form

$$\{p-f_1(x,y)\}\{p-f_2(x,y)\}\ldots\{p-f_n(x,y)\}=0.$$

Then each factor can be equated to zero and the resulting equations of the first degree and first order solved. If the solutions are

$$F_1(x,y,c_1)=0, \quad F_2(x,y,c_2)=0, \ldots, \tag{1}$$

the solution of the given equation can evidently be put in the form

$$F_1(x,y,c)\,F_2(x,y,c)\ldots F_n(x,y,c)=0. \tag{2}$$

There is no loss of generality in replacing the arbitrary constants $c_1, c_2, \ldots, c_n$ by a single arbitrary constant c, because every particular solution obtainable from the equations (1) can also be obtained from (2) giving a suitable value to c.

Ex. 1. Solve $\quad p^2-5p+6=0.$ [*Delhi*, 1959]

Here $\quad p=2$ or 3.

The corresponding solutions are $y=2x+c$, $y=3x+c$.

So the solution of the given differential equation is

$$(y-2x-c)(y-3x-c)=0.$$

Ex. 2. Solve $(x-y)^2p^2-3y(x-y)p+2y^2+xy-x^2=0.$

Solving for p, we get

$$p=\frac{3y(x-y)\pm(x-y)\sqrt{(9y^2-8y^2-4xy+4x^2)}}{2(x-y)^2}$$

$$=\frac{3y\pm(2x-y)}{2(x-y)}=\frac{x+y}{x-y},\ \frac{2y-x}{x-y}.$$

Put now $y=vx$. Taking the first of the two roots for p, we get

$$v+x\frac{dv}{dx}=\frac{1+v}{1-v},\ \text{or}\ c_1+\int\frac{dx}{x}=\int\frac{1-v}{1+v^2}\,dv,$$

or $$\log cx=\tan^{-1}v-\tfrac{1}{2}\log(1+v^2). \quad . \quad . \quad . \quad (1)$$

The second root gives

$$v+x\frac{dv}{dx}=\frac{2v-1}{1-v},\ \text{or}\ c_1+\int\frac{dx}{x}=\int\frac{(1-v)\,dv}{v^2+v-1},$$

or $$\log cx=-\tfrac{1}{2}\log(v^2+v-1)+\frac{3}{2\sqrt{5}}\log\frac{2v+1-\sqrt{5}}{2v+1+\sqrt{5}}. \quad . \quad . \quad . \quad (2)$$

(1) and (2) with v replaced by y/x constitute the required solution.

NOTE. Even if an equation is resolvable into factors which are linear in p, it may not be possible to solve the equation by the method of the present article, for the component differential equations may not be solvable. In such cases the methods of the succeeding articles may be tried.

However, in every case in which the various terms in the differential equation are of the same degree in x, y, as in the last example, the differential equations obtained by solving for p are homogeneous, and so a solution is always possible by this method.

EXAMPLES

Solve the following differential equations :

1. $p^2-2p-3=0$.
2. $(p+y+x)(xp+y+x)(p+2x)=0$.
3. $(p-xy)(p-x^2)(p-y^2)=0$.

4. $x^2p^3+y(1+x^2y)p^2+y^3p=0.$ [*Delhi*, 1960]

5. $p^2+2py\cot x=y^2.$

6. $p(p-y)=x(x+y).$

7. $x\left(\dfrac{dy}{dx}\right)^2+(y-x)\dfrac{dy}{dx}-y=0.$ [*Sagar*, 1954]

8. $y\left(\dfrac{dy}{dx}\right)^2+(x-y)\dfrac{dy}{dx}-x=0.$ [*Allahabad*, 1951]

9. $x^2\left(\dfrac{dy}{dx}\right)^2+xy\dfrac{dy}{dx}-6y^2=0.$ [*Banaras*, 1951]

11·2. Equations solvable for y. If the differential equation can be solved for y and thus put in the form

$$y=f(x, p), \quad \ldots \quad (1)$$

differentiation with respect to x gives an equation of the form

$$p=\phi(x, p, dp/dx).$$

Now this equation is a differential equation in the two variables p and x. It may be possible to obtain its solution, say,

$$F(x, p, c)=0. \quad \ldots \quad (2)$$

The elimination of p between (1) and (2) gives us the required solution.

In case the elimination is not feasible, equations (1) and (2) may be regarded as giving x and y in terms of a parameter p; or, if possible, these equations may be solved and the result expressed in the form

$$x=F_1(p, c),$$
$$y=F_2(p, c).$$

The method of the present article is specially useful for equations in which x is entirely absent.

Ex. Solve $x^2+p^2x=yp$. [*Baroda*, 1959]

Solving for y, we get $y=x^2/p+px$.

Differentiating with respect to x,

$$p=\frac{2xp-x^2p'}{p^2}+p+xp',$$

or
$$\frac{dp}{dx}(p^2-x)+2p=0,$$

i.e.,
$$\frac{dx}{dp}-\frac{x}{2p}=-\tfrac{1}{2}p,$$

which is a linear differential equation. The solution is

$$x=c\sqrt{p}-\tfrac{1}{3}p^2. \quad . \quad . \quad . \quad (1)$$

Substituting this in the original differential equation, we get

$$y=(c\sqrt{p}-\tfrac{1}{3}p^2)^2/p+p(c\sqrt{p}-\tfrac{1}{3}p^2). \quad . \quad . \quad . \quad (2)$$

Equations (1) and (2), which express x and y in terms of p, constitute the solution.

11·3. Equations solvable for x. If the equation can be solved for x and thus can be put in the form

$$x=f(y, p), \quad . \quad . \quad . \quad (1)$$

differentiation with respect to y gives an equation of the form

$$1/p=\phi(y, p, dp/dy).$$

If it is possible to solve this differential equation, let the solution be

$$F(y, p, c)=0. \quad . \quad . \quad . \quad (2)$$

The elimination of p between (1) and (2) gives us the required solution; or x and y may be expressed in terms of p, and p may be regarded as a parameter.

The present method is specially useful for equations in which y is entirely absent.

Ex. Solve the differential equation

$$p^3-4xyp+8y^2=0.$$

Solving for x, we get $x=\dfrac{p^3+8y^2}{4yp}=\dfrac{p^2}{4y}+\dfrac{2y}{p}$.

Differentiating with respect to y,

$$\frac{1}{p}=\frac{pp'}{2y}-\frac{p^2}{4y^2}+\frac{2}{p}-\frac{2yp'}{p^2},$$

or

$$p'\left(\frac{p}{2y}-\frac{2y}{p^2}\right)=\frac{p^2}{4y^2}-\frac{1}{p}, \quad \ldots \quad (1)$$

i.e.,

$$\frac{dp}{dy}=\frac{p}{2y},$$

in which the variables are separable. The solution is

$$p^2=cy.$$

Eliminating p between this and the original differential equation, we get

$$c^{3/2}y^{3/2}-4c^{1/2}xy^{3/2}+8y^2=0, \quad \text{or} \quad \tfrac{1}{2}c^{1/2}(\tfrac{1}{4}c-x)=-y^{1/2}.$$

Hence, writing c for $\frac{1}{4}c$, the required solution is

$$y=c(x-c)^2.$$

NOTE. The significance of the factor $(p^3-4y^2)/yp^2$, by which equation (1) has been divided above, will be discussed later. See § 12·22.

EXAMPLES

Solve the following differential equations :

1. $y=3x+a\log p$.
2. $2y=a^2+x^2+p^2$. [*Banaras, Geophysics*, 1957]
3. $y=\sin p-p\cos p$. [*Nagpur*, 1953]
4. $y=a+bp+cp^2$. [*Nagpur*, 1956]
5. $y=a\sqrt{(1+p^2)}$.
6. $y+xp=x^4p^2$. [*Gujarat*, '62]
7. $xp^3=a+bp$. [*Sagar*, '54]
8. $(2x-b)p=y-ayp^2$.
9. $y^2\log y=xyp+p^2$. [*Allahabad*, 1959]

11·4. Clairaut's equation. (i) The equation

$$y=px+f(p)$$

is known as Clairaut's equation.*

To solve it, differentiate it with respect to x. We get

$$p=p+\{x+f'(p)\}p'.$$

Hence $$p'=0, \text{ or } x+f'(p)=0.$$

The first equation gives

$$p=c.$$

Elimination of p between this and the original differential equation gives the required solution

$$y=cx+f(c).$$

If we eliminate p between

$$x+f'(p)=0$$

and the original equation, we get a solution which does not contain any arbitrary constant, and is not a particular case of the solution $y=cx+f(c)$. Such a solution is called a *singular solution*, and is considered in greater detail in the next chapter.

Sometimes an equation can be reduced to Clairaut's form by a suitable substitution.

(ii) The more general equation

$$y=xf(p)+F(p) \quad . \quad . \quad . \quad (1)$$

*A. C. Clairaut (1713-1765) was a youthful prodigy. He read G. F. de l' Hospital's works on the infinitesimal calculus and on conic sections at the age of ten. Some of his researches were ready for publication at the age of sixteen. In researches on the figure of the earth no other person has accomplished as much as Clairaut. His work on the motion of the moon is equally important. He applied the process of differentiation to solve the differential equation now known by his name. (Cajori, *A History of Mathematics*.)

can be dealt with in a similar way. In fact this equation and also Clairaut's equation are particular cases of the equation considered in § 11·2.

Differentiating (1) with respect to x, we get

$$p=f(p)+\{x f'(p)+F'(p)\}\, dp/dx,$$

or
$$\{p-f(p)\}\frac{dx}{dp}-xf'(p)=F'(p),$$

which is linear, and so can be solved by the method of § 10·5.

Ex. Solve $\qquad y=px+a/p.\qquad$ [*Calcutta*, 1960]

This is of Clairaut's form, and so the solution is

$$y=cx+a/c,$$

where c is an arbitrary constant.

The singular solution is the result of eliminating p between

$$y=px+a/p \quad \text{and} \quad x-a/p^2=0,$$

i.e., the singular solution is

$$y=x\sqrt{(a/x)}+a\sqrt{(x/a)}=2\sqrt{(ax)},$$

or
$$y^2=4ax.$$

Examples

Solve the following differential equations:

1. $y=px+ap(1-p)$. [*Patna*, 1958]
2. $y=px+(1+p^2)^{1/2}$. [*Patna*, 1948]
3. $p=\log(px-y)$. [*Baroda*, 1959]
4. $(y-px)(p-1)=p$. [*Nagpur*, 1954]
5. $xp^2-yp+a=0$. [*Nagpur*, 1956]
6. Use the transformation $x^2=u, y^2=v$ to solve
$(px-y)(py+x)=h^2p$. [*Allahabad*, 1959]
7. Solve $\quad y=2px+y^2p^3$. [*Gorakhpur*, 1960]
[Hint. Multiply by y and put $y^2=v$.]
8. Solve $\quad x^2(y-px)=yp^2$. [*Allahabad*, 1963]
[Hint. Put $x^2=u$, $y^2=v$.]

EXAMPLES ON CHAPTER XI

Solve

1. $y=x(dy/dx)+(dy/dx)^3$. [*Banaras, Geophysics*, '61]
2. $y=xp^2+p$. [*Delhi*, 1951]
3. $y=2px+p^n$. [*Nagpur*, 1953]
4. $x^2\left(\dfrac{dy}{dx}\right)^2-2xy\dfrac{dy}{dx}+2y^2-x^2=0$. [*Aligarh*, 1960]
5. $y=(p+p^n)x+1/p^{n-1}$. [*Nagpur*, 1952]
6. $y=p^2y+2px$. [*Sagar*, 1951]
7. $y=x\{p+\sqrt{(1+p^2)}\}$. [*Sagar*, 1950]
8. $axyp^2+(x^2-ay^2-b)p-xy=0$. [*Allahabad*, 1960]
9. $p^2(x^2-a^2)-2pxy+y^2+a^2=0$.
10. $y=apx+bp^3$.
11. $xyp^2+p(3x^2-2y^2)-6xy=0$. [*Banaras, Geophysics*, '60]
12. $x^2p^2-2xyp+y^2=x^2y^2+x^4$.
13. $4(xp^2+yp)=y^4$. [*P.S.C., U.P.*, 1954]
14. $y-2px=f(xp^2)$. [*Allahabad*, 1959]
15. $9(y+xp\log p)=(2+3\log p)p^3$. [*P.S.C., U.P.*, 1955]
16. $y-x=x\dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^2$.
17. $x+p/\sqrt{(1+p^2)}=a$. [*Banaras*, 1950]
18. $p^3-p(y+3)+x=0$. [*P.S.C., U.P.*, 1952]
19. $(x-a)p^2+(x-y)p-y=0$.
20. $p=\tan\{x-p/(1+p^2)\}$. [*Poona*, 1957]
21. $e^{3x}(p-1)+p^3e^{2y}=0$. [*Gujarat*, 1962]
22. $(y-xp)^2/(1+p^2)=a^2$. [*Banaras*, 1960]
23. Find the general and singular solutions of the differential equation $(xp-y)^2=p^2-1$.
24. By differentiating with respect to x the equation

$$p^3+xp^2=y,$$

obtain its general solution in the form $x=f(p), y=\phi(p)$. [*Utkal*, 1947]

CHAPTER XII

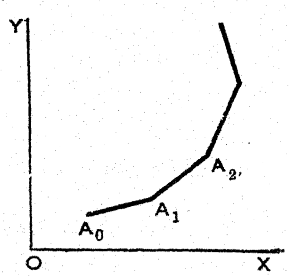
GEOMETRICAL INTERPRETATION. APPLICATIONS

12·1. Geometrical meaning of a differential equation. Consider first the differential equation of the first order and the first degree,

$$f(x, y, dy/dx)=0. \quad . \quad . \quad . \quad (1)$$

Let x, y denote the coordinates of a point. Then the differential equation (1) can be regarded as an equation giving the value of dy/dx when the values of x and y are known.

Let A_0 be any point (a_0, b_0). Let the value of p $(\equiv dy/dx)$ at (a_0, b_0) derived from the equation (1) be p_0. Take a point A_1 at a short distance from A_0 and in such a direction that the gradient of A_0A_1 (i.e., the tangent of the angle which A_0A_1 makes with the x-axis) is p_0. Let the coordinates of A_1 be (a_1, b_1) and let the corresponding value of p be p_1. Take A_2 at a short distance from A_1 in such a direction that the gradient of A_1A_2 is p_1, and so on.



Let now A_0A_1, A_1A_2, ... tend to zero (and let their number tend to infinity); and suppose that the broken curve $A_0A_1A_2$ ... tends to a curve C. Then the curve C evidently possesses the property

that the gradient of the tangent to it at any point and the coordinates of that point satisfy the differential equation (1).

If we start with any other point, different from A_0, and not lying on the curve C, we shall obtain another curve which also possesses the above property. Evidently we can obtain a family of curves each member of which possesses the above property.

Moreover, we have seen that the solution of the differential equation (1) is of the form

$$F(x, y, c)=0,$$

where c is an arbitrary constant. This also shows that the differential equation represents a family of curves.

Hence the differential equation $f(x, y, dy/dx)=0$ represents a family of curves each member of which possesses the property that at any point (x, y) on it the value of dy/dx and the coordinates x, y satisfy the differential equation.*

The differential equation of the first order, but of the second degree, is a quadratic in dy/dx and so gives us two values of dy/dx at every point.

*The broken curve $A_0A_1A_2$ ... obtained above will be a very close approximation to the curve given by the differential equation if A_0A_1, A_1A_2, ... are sufficiently small. A numerical solution, corresponding exactly to the above geometrical process, was used to solve the differential equations of motion of Halley's comet and thus predict its return in 1759, because the analytical solution would have taken too much time. Many other equations, which cannot be solved by analytical methods, but whose solution is important for certain researches, have been solved by similar approximate methods.

Hence two members of the family of curves represented by it pass through every point. We may expect, therefore, that its solution will involve the arbitrary constant c in the second degree, so that there may be two values of c for every point. We have already seen in the previous chapter that this in fact is the case.

The method of finding the differential equation when the primitive is given has been explained in §10·11.

Ex. Obtain the differential equation of all circles which have their centre at (a, b). What is the geometrical interpretation of the differential equation?

The Cartesian equation of all circles which have their centre at (a, b) is

$$(x-a)^2+(y-b)^2=c, \qquad . \quad . \quad . \quad (1)$$

where c is arbitrary.

To obtain the differential equation, we differentiate (1). We get

$$(x-a)+(y-b)\,dy/dx=0.$$

As this equation does not contain c, this is the required differential equation.

It can be written as

$$\frac{dy}{dx}=-\frac{x-a}{y-b}.$$

Now the "m" (i.e., the gradient) of the straight line joining (a, b) to (x, y) is $(y-b)/(x-a)$.

So the above differential equation means that at every point the curves represented by it are perpendicular to the line joining that point to (a, b). In other words, the differential equation states that the curves represented by it are the curves which cut the radii vectores from (a, b) at right angles.

EXAMPLES

1. Find the differential equation satisfied by x, y independently of the values of a, b in the equation

$$y = ax \cos\left(\frac{n}{x} + b\right).$$

2. Form the differential equation of the system of circles touching the y-axis at the origin. [*Nagpur*, 1956]

3. Find the differential equation of all coaxal parabolas.

4. Show that the differential equation of a general parabola is

$$\frac{d^2}{dx^2}\left\{\left(\frac{d^2y}{dx^2}\right)^{-2/3}\right\} = 0. \qquad [\textit{I.A.S.}, 1958]$$

5. Form the differential equation of all conics whose axes coincide with the axes of coordinates. [*Aligarh*, 1954]

6. Discuss the graph of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

7. Find the equation of the curve through the origin which satisfies the differential equation

$$dy/dx = (x-y)^2.$$

12·2. Singular solution. Geometrical meaning. Consider the differential equation

$$y = px + a/p, \qquad \ldots \quad (1)$$

which was solved in § 11·4.

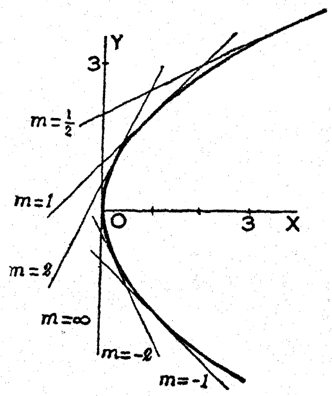
The solution is

$$y = mx + a/m, \qquad \ldots \quad (2)$$

where m is an arbitrary constant, and represents a family of straight lines.

It is well known that the envelope of this family is the parabola

$$y^2=4ax.$$



Now take any point (a_0, b_0) on this parabola and consider first the straight line L_0 of the family (2) which passes through (a_0, b_0). By §12·1 it is evident that the values of x, y, dy/dx at (a_0, b_0) for the straight line L_0 must satisfy the differential equation (1).

Next consider the value, at (a_0, b_0), of dy/dx for the parabola $y^2=4ax$. This value must be the same as the value of dy/dx for the tangent to it at (a_0, b_0), i.e., the same as the value of dy/dx at (a_0, b_0) for the straight line L_0.

The question to be considered now is whether the values, at (a_0, b_0) of x, y, dy/dx for the parabola will satisfy the differential equation (1). The answer evidently must be in the affirmative, because the values of x, y, dy/dx for the parabola belong also to that member of the family of straight lines represented by the differential equation (1) which passes through (a_0, b_0).

We see, therefore, that at (a_0, b_0), and so at every point on the parabola $y^2=4ax$, the values of x, y and dy/dx satisfy the differential equation (1).

Hence $y^2=4ax$ must also be a solution of the differential equation (1).

This solution, which is not a particular case of the general solution (2), is the singular solution, and represents the envelope of the family of curves given by the differential equation.

The above proposition is a general one. *Whenever an envelope of the family of curves represented by the general solution of a differential equation exists, the equation of the envelope is also a solution of the differential equation.* For, if the general solution of the differential equation

$$f(x, y, dy/dx)=0, \quad . \quad . \quad . \quad (3)$$

is

$$F(x, y, c)=0, \quad . \quad . \quad . \quad (4)$$

and the envelope of (4) is

$$\phi(x, y)=0, \quad . \quad . \quad . \quad (5)$$

the values of $x, y, dy/dx$ at any point (a_0, b_0) on (5) must be the same as the values of these quantities at that point for the member of the family (4) which passes through (a_0, b_0), and so must satisfy (3).

The equation of the envelope of the family of curves represented by the general solution of a differential equation is called the *singular solution.* Such a solution does not involve any arbitrary constant. Usually it is not included in the general solution. It may, however, in exceptional cases be only a particular case of the general solution; then it is regarded as being both a singular solution and a particular case of the general solution.

NOTE. We can easily verify directly that $y^2=4ax$ is a solution of the differential equation (1). For $y^2=4ax$ gives

$$dy/dx=2a/y.$$

With this value of p the right-hand side of the differential equation

$$=\frac{2ax}{y}+\frac{ay}{2a}=\frac{\frac{1}{2}y^2}{y}+\tfrac{1}{2}y=y=\text{the left-hand side.}$$

Thus the differential equation is satisfied.

12·21. The singular solution of Clairaut's equation. We have seen that the general solution of Clairaut's equation

$$y=px+f(p) \qquad . \quad . \quad . \quad (1)$$

is

$$y=cx+f(c). \qquad . \quad . \quad . \quad (2)$$

Hence the singular solution, which is the envelope of (2), will be obtained by eliminating c between (2) and the equation

$$0=x+f'(c), \qquad . \quad . \quad . \quad (3)$$

obtained by differentiating (2) partially with respect to c.

Now the equation obtained by differentiating (1) partially with respect to p is

$$0=x+f'(p). \qquad . \quad . \quad . \quad (4)$$

The equations (1) and (4) differ from the equations (2) and (3) only in having p instead of c. Hence the result of eliminating p between (1) and (4) will be the same as that of eliminating c between (2) and (3), i.e., the equation, obtained by eliminating p between (1) and (4) will be the envelope of the curves (2), and thus will be the singular solution of (1).

By comparison with § 11·4 it will be seen that equation (4) was one of the equations obtained on differentiating (1) with respect to x. The above shows why the elimination of p between that equation and the differential equation gave the singular solution.

12·22. Determination of singular solutions. General case. An equation which possesses a singular solution is not considered completely solved until the singular solution also has been found. Hence it is necessary to know how to find such a solution.

There are two methods. We may find the singular solution either from the differential equation or from its general solution. Since the singular solution is the envelope of the family of curves represented by the general solution, it can be found from the general solution by the usual method of finding envelopes. Thus *if the general solution is*

$$F(x, y, c)=0, \qquad . \quad . \quad . \quad (1)$$

the singular solution is obtained by eliminating c between it and the equation

$$\partial F(x, y, c)/\partial c=0. \qquad . \quad . \quad . \quad (2)$$

Again, if the result of this elimination be

$$\phi(x, y)=0, \qquad . \quad . \quad . \quad (3)$$

it is well known that this equation represents the condition that two roots of (1), considered as an equation in c, should be equal. Geometrically interpreted, this means that the condition that the point $P(x, y)$ should lie on the envelope (3) is the condition that two of the curves, of the family (1), which pass through (x, y) should coincide.

Regarded from this point of view, it is obvious that the equation of the envelope should also be obtainable from the condition that the values of p for two of the curves which pass through (x, y) should be equal; i.e., *if the differential equation is*

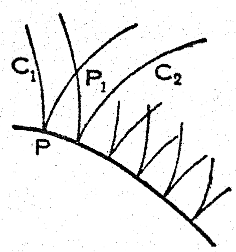
$$f(x, y, p)=0,$$

the envelope, and hence the ***singular solution***, *will be obtained by eliminating p between it and*

$$\partial f(x, y, p)/\partial p = 0.$$

The above makes it evident that *if p occurs only in the first degree in the differential equation, there will be no singular solution.* Similarly, if the differential equation can be resolved into a number of factors, each linear in p, there will be no singular solution.

The process of finding envelopes in some cases gives us curves which are not envelopes (see *Text-Book on Diff. Cal.*, § 12·21). The same is true about the process of finding the singular solution. The process will give, for example, the locus of cusps if each member of the family possesses a cusp. But the dy/dx for this locus will in general be quite different from the dy/dx for each member, as in the accompanying figure; i.e., its equation will not satisfy the differential equation. Therefore, in any particular case, unless the equation obtained for the singular solution obviously represents the envelope and nothing but the envelope, it is necessary to try whether the result satisfies the differential equation. Should it not do so, it may happen that the equation can be resolved into others that are simpler, and one or more than one of them may satisfy the equation; these will then constitute the singular solution.*



NOTE 1. In solving differential equations by the methods of §§11·2, or 11·3, the equation obtained after the differentiation with respect to x or y often contains a factor which, equated to zero, gives what we would obtain by differentiating the given differential equation with respect

*A. R. Forsyth, *A Treatise on Differential Equations*, where a more detailed treatment of singular solutions will be found than is possible to give here.

to p. Therefore the elimination of p between this and the original equation will, in general, give us the singular solution.

NOTE 2. If $F(c)=0$ is an algebraic equation in c, the simplest function of the coefficients the vanishing of which represents the condition that the equation should have two equal roots is called the *discriminant*. Thus the discriminant of the equation

$$Ac^2+Bc+C=0 \qquad \ldots \quad (4)$$

is

$$B^2-4AC.$$

The equation $B^2-4AC=0$ may be called the *discriminant* relation.

In confirmation of what has been said above about the envelope of a family of curves being the locus of a point for which the two members of the family passing through it coincide, it may be noted that if A, B, C are functions of x, y, the envelope of the family of curves (4) is $B^2-4AC=0$, and this equation is also the condition that the quadratic (4) should have two equal roots.

Ex. Solve completely the differential equation

$$9p^2(2-y)^2=4(3-y). \qquad \text{[\textit{Lucknow}, 1958]}$$

Here

$$3p=3\frac{dy}{dx}=\pm\frac{2\sqrt{(3-y)}}{2-y}.$$

The variables are separable. Integration gives

$$x+c=\pm\tfrac{3}{2}\int\frac{(2-y)\,dy}{\sqrt{(3-y)}}=\mp\tfrac{3}{2}\int\frac{(t^2-1)2t\,dt}{t},$$

where $t=\sqrt{(3-y)}$,

$$=\mp t(t^2-3),$$

or

$$(x+c)^2=y^2(3-y).$$

This is the general solution. The singular solution can be found by any of the two methods given below.

First method. The general solution is

$$(x+c)^2=y^2(3-y).$$

This is a quadratic in the parameter c. Hence the envelope is

$$x^2-\{x^2-y^2(3-y)\}=0,$$

i.e.,

$$y^2(3-y)=0.$$

Now $y=0$ gives $p=0$. Substitution in the differential equation shows that these values of y and p do not satisfy it. Hence $y=0$ is not a solution.

Again $3-y=0$ gives $p=0$. Substitution shows that these values of y and p satisfy the differential equation. Hence the singular solution is $y=3$.

Second method. The given differential equation is a quadratic in p. So the p-discriminant relation can be written down at once. It is

$$144(2-y)^2(3-y)=0.$$

Now $2-y=0$ gives $p=0$. Substitution in the differential equation shows that these values of y and p do not satisfy it. Hence $2-y=0$ is not a solution.*

Again, $3-y=0$ gives $p=0$. Substitution shows that these values of y and p do satisfy the differential equation. Hence the singular solution is $y=3$.

Examples

Find the general and singular solutions of

1. $(y-px)^2+a^2p=0$.　　2. $3xy=2px^2-2p^2$.

3. $p^3-4xyp+8y^2=0$.　　[*Patna*, 1947]

4. $(y-px)^2(1+p^2)=a^2p^2$.

5. $y-px+x-y/p=a$.

6. Find the complete primitive and singular solution of

$$y=px+\sqrt{(b^2+a^2p^2)}.$$

Interpret your results geometrically.　　[*Banaras*, 1956]

7. Investigate for singular solutions

$$4x(x-1)(x-2)p^2-(3x^2-6x+2)^2=0.$$

8. Find the general and singular solutions of

$$y^2-2pxy+p^2(x^2-1)=m^2.$$　　[*Lucknow*, '59]

*For the geometrical meaning of the various loci see Miller: *A First Course in Differential Equations*, p. 31.

12·3. Geometrical problems. Many geometrical problems give rise to differential equations. An example will make the procedure to be adopted in solving such problems clear.

Ex. Find the curves in which the polar subnormal is of constant length.

Let the length of the polar subnormal be a. Then, since the polar subnormal is given by the expression $dr/d\theta$, we must have

$$dr/d\theta = a,$$

which is the differential equation of the required curves.

Solving it, we get

$$r = a(\theta + c),$$

where c is an arbitrary constant. This is the polar equation of the required curves.

Examples

1. Find the Cartesian equation of the curve whose subtangent is constant.

2. Find the curve in which the polar subtangent is constant.

3. Find the curve in which the subnormal is equal to the abscissa.

4. Show that the parabola is the only curve in which the subnormal is constant. [*Delhi*, 1950]

5. Find the differential equation of the family of curves which cut a family of coaxal circles at a constant angle.

6. Find the Cartesian equation of the curve in which the perpendicular from the foot of the ordinate on the tangent is of constant length. [*Madras*, 1950]

7. Find the curves for which the sum of the reciprocals of the radius vector and the polar subtangent is constant. [*Agra*, 1956]

8. Show that the curve in which the angle between the tangent and the radius vector at every point is one-half of the vectorial angle is a cardioid. [*Agra*, 1958]

9. Find the curve which is such that the portion of the x-axis cut off between the origin and the tangent at any point, is proportional to the ordinate of the point. [*Del.*, '62]

10. The slope of a curve at any point is the reciprocal of twice the ordinate at that point. The curve also passes through the point (4, 3). Find the equation to the curve. [*Andhra*, 1940]

11. The tangent at a point P of a curve meets the axis of y at M and the parallel through P to the axis of y meets the axis of x at N. O is the origin. If the area of the triangle MON is constant, show that the curve is a hyperbola. [*Lucknow*, 1960]

12. Show that the curve for which the radius of curvature varies as the square of the perpendicular upon the normal belongs to the class whose pedal equation is

$$r^2-p^2=p/k+1/2k^2+Ae^{2kp},$$

k being a given constant and A arbitrary.

13. By integrating twice, or otherwise, find the primitive of

$$(1+x^2)\frac{d^2u}{dx^2}+2x\frac{du}{dx}=0.$$

Hence, or otherwise, obtain the nth derivative of $\tan^{-1}x$ at $x=0$.

12·4. Trajectories. A curve which cuts every member of a given family of curves in accordance with some given law is called a *trajectory* of the given family of curves. We shall consider only the case when the given law is that the angle at which the curve cuts every member is constant.

If a curve cuts every member of a given family of curves at right angles, it is called an *orthogonal*

trajectory. The orthogonal trajectories of a given family of curves themselves form a familiy of curves. Their differential equation is easy to find if the differential equation of the original family of curves is known. For, let the given family of curves have the equation

$$f(x, y, dy/dx)=0, \quad . \quad . \quad . \quad (1)$$

and suppose that X, Y are the current coordinates of any point on an orthogonal trajectory of (1).

At the point where a member of (1) cuts the orthogonal trajectory, we must have

$$X=x,$$
$$Y=y,$$
$$\frac{dY}{dX}=-\frac{1}{dy/dx}.$$

Substituting in (1), we get

$$f(X, Y, -dX/dY)=0,$$

which is the required differential equation of the orthogonal trajectories.

Thus *to obtain the differential equation of the orthogonal trajectories, we have simply to write* $-dx/dy$ *for* dy/dx *in the differential equation of the original family of curves.*

Similarly, since the tangent to a curve makes with the radius vector an angle ϕ, where

$$\tan\phi=r\,d\theta/dr,$$

the tangent to the orthogonal trajectory must make with the radius vector an angle Φ, where

$$\tan\Phi=R\,d\Theta/dR,$$

(R, Θ) being any point on the trajectory. But

$$\Phi=\phi+\tfrac{1}{2}\pi.$$

Hence $\qquad (r\,d\theta/dr)(R\,d\Theta/dR)=-1.$

It follows that *the differential equation of the orthogonal trajectories is obtained from the differential equation (in polar coordinates) of a given family of curves by writing*

$$-\frac{1}{r}\frac{dr}{d\theta} \quad \text{for} \quad r\frac{d\theta}{dr},$$

i.e.,
$$-r^2\frac{d\theta}{dr} \quad \text{for} \quad \frac{dr}{d\theta}.$$

If the trajectories cut the given family of curves, whose differential equation is

$$f(x, y, dy/dx)=0,$$

at the constant angle α, instead of at right angles, the difference between

$$\tan^{-1}\left(\frac{dy}{dx}\right) \quad \text{and} \quad \tan^{-1}\left(\frac{dY}{dX}\right)$$

must be α, i.e.,

$$\frac{dy}{dx}=\frac{dY/dX-\tan\alpha}{1+(dY/dX)\tan\alpha}, \quad \text{or} \quad \frac{dY/dX+\tan\alpha}{1-(dY/dX)\tan\alpha};$$

according as the trajectories make the angle α on one side or the other of the curve.

Hence the two differential equations of the trajectories are obtained by substituting for dy/dx in the differential equation of the given family of curves, the expressions

$$\frac{dy/dx-\tan\alpha}{1+(dy/dx)\tan\alpha} \quad \text{and} \quad \frac{dy/dx+\tan\alpha}{1-(dy/dx)\tan\alpha}$$

respectively.

To obtain the trajectories themselves, their differential equation obtained by any of the above methods must be integrated. When only the ordinary equation of the original family of curves is known, its differential equation must first be found by differentiation and elimination of the parameter.

Ex. Find the orthogonal trajectories of the cardioids $r=a(1-\cos\theta)$, a being the parameter. [*I.A.S.*, 1960]

Differentiating $r=a(1-\cos\theta)$,

we get $dr/d\theta=a\sin\theta$.

Eliminating a, we have $r\,d\theta/dr=(1-\cos\theta)/\sin\theta$.

Hence the orthogonal trajectories are given by the equation

$$-\frac{1}{r}\frac{dr}{d\theta}=\frac{1-\cos\theta}{\sin\theta}.$$

Separating the variables and integrating, we have

$$\int\frac{dr}{r}=-\int\frac{(1-\cos\theta)\,d\theta}{\sin\theta},$$

or $$\log r=-\log\tan\tfrac{1}{2}\theta+\log\sin\theta+\log c,$$

i.e., $$r=c\sin\theta/\tan\tfrac{1}{2}\theta=2c\cos^2\tfrac{1}{2}\theta.$$

Hence $$r=c(1+\cos\theta)$$

is the required family of orthogonal trajectories.

EXAMPLES

Find the equation of the family of curves that is orthogonal to

1. $y=ax^2$. [*Vikram*, '61] 2. $y=(x^3-a^3)/3x$.

3. $ax^2+y^2=1$. 4. $xy=k^2$. [*Delhi*, 1962]

5. Show that the orthogonal trajectories of the family of conics

$$y^2-x^2+4xy-2cx=0$$

consist of a family of cubics with the common asymptote $x+y=0$. [*P.S.C., U.P.*, 1960]

6. Find the differential equation satisfied by the system of parabolas $y^2=4a(x+a)$, and show that the orthogonal trajectories of the system belong to the system itself. [*Delhi*, '60]

7. Find the orthogonal trajectories of the system of curves

$$\left(\frac{dy}{dx}\right)^2=\frac{a}{x}.$$

8. Prove that the families of curves given by the equations

$$y^2+3x^2=2ax,$$
$$y^3=b(y^2-x^2),$$

where a and b are arbitrary parameters, intersect at right angles.

Find the orthogonal trajectories of

9. $r\theta=a$.
10. $r=a\theta$. [*Banaras*, '41]
11. $r=e^{a\theta}$.
12. $r=a(1+\cos n\theta)$.
13. $r^n=a^n \sin n\theta$.
14. $r^n \sin n\theta=a^n$. [*Banaras*, '59]

Find the equation of the family of oblique trajectories which cut

15. A family of concentric circles at 30°.
16. The straight lines $y=mx$ at 45°.
17. The circles touching the x-axis at the origin at 60°.

12·5. Other applications. Differential equations are of great utility in many problems of mechanics and physics. Such problems, however, are not considered here, as the student is sure to study them in connection with their respective subjects.

EXAMPLES ON CHAPTER XII

1. Find the differential equation of all circles in the (x, y) plane.

2. Find the differential equation of a system of confocal and coaxal parabolas. [*Travancore*, 1958]

3. Show that the differential equation of all hyperbolas passing through the origin and having their asymptotes parallel to the coordinate axes is

$$xy\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2+2y\frac{dy}{dx}=0.$$

4. Prove that, through any point (x, y) for which $x\neq0$, there passes exactly one curve satisfying the differential equation $dy/dx=ky/x$,

where k is a given constant. Prove that, if $k>0$, all these curves pass through the origin. Prove that, if $k<0$, the only one of the curves which passes through the origin is the x-axis.

5. Find the curves in which the subtangent varies as the abscissa. [*Rajasthan*, 1957]

6. Find the equation of the curve for which the polar subnormal varies as the radius vector. [*Delhi*, 1962]

7. Find the curves for which the Cartesian subnormal varies as the square of the radius vector. [*Bihar*, 1956]

8. Find the polar equation of the curves in which the length of arc is proportional to the vectorial angle.

9. Find the equation to the family of curves in which the length of the tangent between the point of contact and the x-axis is of constant length equal to a. [*Nagpur*, 1956]

10. Obtain the Cartesian equation of the curve which possesses the property that the rectangle contained by the radius vector and the perpendicular drawn from the origin to the tangent is a constant $(=k^2)$; given that the curve cuts the x-axis at a distance k from the origin.

11. Find the curves in which the projection of the radius of curvature on the axis of y has a constant value a.

12. The tangent at any point P of a curve meets the x-axis at Q. If Q is on the positive side of the origin O and $OP=OQ$, show that the family of curves having this property are parabolas whose common axis is the x-axis.

Find the equation of the family of orthogonal trajectories. [*Allahabad*, 1959]

13. The normal at any point P of a curve cuts the x-axis in G, and N is the foot of the ordinate of P. If NG varies as the square of the radius vector from the origin, find the differential equation to the curve, and solve it. [*Alld.*, '60]

14. Solve and examine for singular solution

$$x^3p^2+x^2yp+a^3=0. \qquad [\textit{Gujarat}, 1957]$$

15. Solve

$$x^2p^2+yp(2x+y)+y^2=0,$$

and obtain the singular solution. [*Lucknow*, 1962]

16. Find the general and singular solutions of

$$27y-8p^3=0.$$

17. Find the orthogonal trajectories of a family of circles which touch a given line at a given point. [*Gorakhpur*, '60]

18. A system of rectangular hyperbolas pass through the fixed points $(\pm a, 0)$ and have the origin as centre; show that the orthogonal trajectories are given by

$$(x^2+y^2)^2=2a^2(x^2-y^2)+C.$$ [*Utkal, Hons.*, '46]

19. Show that the system of confocal conics

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

is self-orthogonal. [*Lucknow*, 1962]

20. Find the orthogonal trajectories of the family of semicubical parabolas $ay^2=x^3$, where a is a variable parameter. [*Delhi*, 1959]

21. Find the differential equation of the family of curves given by the equation $x^2-y^2+2\lambda xy=1$, where λ is a parameter. Obtain the differential equation of its orthogonal trajectories and solve it. [*Lucknow*, 1944]

22. Prove that the orthogonal trajectories of the curves

$$A=r^2\cos\theta$$

are the curves

$$B=r\sin^2\theta.$$

23. Show that the orthogonal trajectories of

$$x^n/a^n+y^n/b^n=c$$

are

$$a^nx^{2-n}-b^ny^{2-n}=c',$$

where c and c' are variable parameters.

Examine the cases $n=1$, and $n=2$.

24. Find the trajectories orthogonal to $y=\tan x+c$, and illustrate the families of curves in a sketch.

25. Find the equation of the system of orthogonal trajectories of a system of confocal and coaxal parabolas

$$r=2a/(1+\cos\theta).$$ [*Poona*, 1960]

CHAPTER XIII

LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

13·1. Definitions. We have already defined a linear differential equation as an equation in which the dependent variable y and its differential coefficients occur only in the first degree. If, further, the coefficient of y and those of its differential coefficients are constants, the equation is said to be a linear differential equation with constant coefficients. The form of such an equation, therefore, is

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_n y=Q, \quad (1)$$

where Q is any function of x.

Consider first the equation in which the second member, viz. Q, is zero:

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_n y=0. \quad (2)$$

Substitution will show at once that the following properties are true for this equation:

(i) If $y=f_1(x)$ is a solution, then $y=cf_1(x)$, where c is an arbitrary constant, is also a solution.

(ii) If $y=f_1(x)$, $y=f_2(x)$, ..., $y=f_n(x)$ are solutions, then,

$$y=c_1 f_1(x)+c_2 f_2(x)+\ldots+c_n f_n(x),$$

where c_1, c_2, ..., c_n are arbitrary constants, is also a solution.

Now if the n functions $f_1(x), f_2(x), \ldots, f_n(x)$ are linearly independent,* the solution

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x),$$

which contains n arbitrary constants, must be the general solution of the equation under consideration, which is of the nth order. (If $f_1(x), f_2(x), \ldots$ are not linearly independent, some of the terms can be combined and the number of arbitrary constants left will be less than n.)

Next consider equation (1), in which the left-hand member is the same as in equation (2), but the right-hand member is different from zero. Substitution shows at once that if $c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x)$ is a solution of (2), and $\phi(x)$ is any particular solution of (1), then

*If we can find constants $b_1, b_2, \ldots, b_n$ such that

$$b_1 f_1(x) + b_2 f_2(x) + \ldots + b_n f_n(x) \equiv 0,$$

then the functions $f_1(x), f_2(x), \ldots$ are not linearly independent, for we can express any one of them linearly in terms of the others. Thus

$$f_1(x) \equiv -(b_2/b_1) f_2(x) - (b_3/b_1) f_3(x) - \ldots.$$

If, on the other hand, no set of constants $b_1, b_2, \ldots, b_n$ exists such that

$$b_1 f_1(x) + b_2 f_2(x) + \ldots + b_n f_n(x) \equiv 0,$$

we say that the functions $f_1(x), f_2(x), \ldots$ are *linearly independent*. We add the adverb 'linearly' because two functions of x are never independent of each other in the general sense. Take, for example, $\sin x$ and x^3. Given x^3 we can find x and therefore $\sin x$. Hence $\sin x$ can be expressed as a function of x^3; in fact, $\sin x \equiv \sin\{(x^3)^{1/3}\}$, the real cube root being taken. But $\sin x$ and x^3 are linearly independent. On the other hand, $x^2 - x^3$, $x^2 + x^3$ and $x^2 + 2x^3$ are not linearly independent, because $x^2 + 2x^3 \equiv -\frac{1}{2}(x^2 - x^3) + \frac{3}{2}(x^2 + x^3)$.

$$c_1f_1(x)+c_2f_2(x)+\ldots+c_nf_n(x)+\phi(x)$$

is also a solution of (1).

If $f_1(x), f_2(x), \ldots$ are linearly independent, this must be the most general solution, as it contains the full number of arbitrary constants.

Thus the general solution of (1) consists of two parts, one of which contains n arbitrary constants and is a solution of the equation obtained from (1) by putting the second member equal to zero, and the other contains no arbitrary constants. The former is called the *complementary function* and the latter the *particular integral.*

13·2. Second member zero. Roots of the auxiliary equation all different. Assume, tentatively, that e^{mx} is a solution of equation (2) of the previous article. Substitution shows that we must have

$$e^{mx}(m^n+a_1m^{n-1}+a_2m^{n-2}+\ldots+a_n)=0.$$

Hence e^{mx} will be a solution of (2) if m is a root of the algebraic equation

$$m^n+a_1m^{n-1}+a_2m^{n-2}+\ldots+a_n=0,$$

which is called the *auxiliary equation.*

If the roots of this are m_1, m_2, ..., m_n, and they are all different, e^{m_1x}, e^{m_2x}, ... are all different and linearly independent. So the general solution of (2) in the case is

$$c_1e^{m_1x}+c_2e^{m_2x}+\ldots+c_ne^{m_nx}.$$

Ex. Solve the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}-4y=0. \qquad \text{[Calcutta, 1960]}$$

The auxiliary equation is $m^2-3m-4=0$,

i.e.,
$$(m-4)(m+1)=0.$$

The roots are -1 and 4. Hence the required solution is

$$y=c_1e^{-x}+c_2e^{4x}.$$

EXAMPLES

1. Show by actual substitution that

(i) e^x is a solution of the differential equation

$$\frac{d^2y}{dx^2}-y=0,$$

(ii) e^x+c is not a solution,

(iii) e^{-x} is a solution,

(iv) ae^x+be^{-x} is also a solution,

(v) ae^x+be^{-x} is not a solution of the equation

$$\frac{d^2y}{dx^2}-y=\sin x,$$

but (vi) $e^x-\frac{1}{2}\sin x$ is a solution,

(vii) $e^{-x}-\frac{1}{2}\sin x$ is also a solution,

(viii) $ae^x+be^{-x}-\frac{1}{2}\sin x$ is also a solution,

(ix) $ae^x+be^{-x}+c-\frac{1}{2}\sin x$ is not a solution.

What is the general solution of $d^2y/dx^2-y=\sin x$? Which part of it is the complementary function and which the particular integral ?

Solve

2. $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=0.$

3. $\dfrac{d^2y}{dx^2}-7\dfrac{dy}{dx}+12y=0.$

4. $\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}+4y=0.$

5. $\dfrac{d^3y}{dx^3}-6\dfrac{d^2y}{dx^2}+11\dfrac{dy}{dx}-6y=0.$

6. $\dfrac{d^3y}{dx^3}-13\dfrac{dy}{dx}+12y=0.$

7. $\dfrac{d^3y}{dx^3}-9\dfrac{d^2y}{dx^2}+23\dfrac{dy}{dx}-15y=0.$

8. $\dfrac{d^2x}{dt^2}-3\dfrac{dx}{dt}+2x=0,$

given that when $t=0$, $x=0$ and $dx/dt=0$.

13·3. The symbol D. There is a special convenience in using the symbols D and D^n for

$$\frac{d}{dx} \text{ and } \frac{d^n}{dx^n}$$

respectively in the treatment of linear differential equations with constant coefficients. In the first place, y may be written only once, when there are a number of terms involving y, Dy, D^2y, ..., by making use of brackets. Thus the equation (2) of § 13·1 may be written as

$$(D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots+a_{n-1}D+a_n)y=0.$$

Again, as will be proved below, D can be treated as an algebraical symbol in several respects. This greatly facilitates the solution of the differential equation.

The meaning of an expression like

$$(D-\alpha)(D-\beta)y$$

is that it represents $(D-\alpha)(Dy-\beta y)$,

i.e., $$D(Dy-\beta y)-\alpha(Dy-\beta y).$$

Thus
$$(D-1)(D-2)e^{3x}=(D-1)(De^{3x}-2e^{3x})$$
$$=(D-1)(3e^{3x}-2e^{3x})$$
$$=D(3e^{3x}-2e^{3x})-(3e^{3x}-2e^{3x}).$$

Similarly in
$$(D-a_1)(D-a_2)(D-a_3)\ldots(D-a_n)y,$$
we must begin with the factor which is next to y, find the result of performing the operation indicated by it, then take the next factor, perform the operation indicated by it, and so on.

We shall prove now that the factors $D-a_1$, $D-a_2$, ... can be written in any order, and the final result will be the same, provided a_1, a_2, ... are constants; moreover, if $D^n+a_1D^{n-1}+\ldots+a_n$ when factorised is equal to $(D-a_1)(D-a_2)\ldots(D-a_n)$, the final value of
$$(D-a_1)(D-a_2)\ldots(D-a_n)y,$$
when the operations indicated by the factors have been performed, will be the same as that of
$$(D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots+a_n)y.$$
To prove this, we notice that the symbol D obviously obeys all the fundamental laws of algebra; for
$$(D^p+D^q)y=D^py+D^qy$$
as well as

$D(y+z)=Dy+Dz$, the Distributive Law;

$D^pD^qy=D^qD^p\,y$, the Commutative Law;

and $D^pD^qy=D^{p+q}y$, the Index Law;

except that the Commutative Law is not true with respect to variables.

Thus $$Dy \neq yD,\ D(yz) \neq y\,Dz.$$

In fact, yD has no meaning; and although $D(yz)$ and $y\,Dz$ both have meanings, they are not equal if y and z are functions of x. For $D(yz) = yDz + zDy$.

It follows that *we are justified in breaking up*

$$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n$$

in the expression

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y$$

into factors and in taking the factors in any order, provided $a_1, a_2, \ldots, a_n$ are constants; for each step can be justified by one or more of the above laws, exactly as in algebra.

Let now $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n$ be denoted by $f(D)$ and let $f_1(D)$ be a factor of $f(D)$, so that $f(D) = f_1(D)\, f_2(D)$. We notice that if y_1 is a solution of the differential equation

$$\{f_1(D)\}y = 0,$$

then it is also a solution of the differential equation

$$f(D)\,y = 0;$$

for, by what we have established above,

$$f(D)\,y \equiv f_2(D)\, f_1(D)\,y,$$

which shows that if $f_1(D)y$ is zero, then $f(D)y$ also must be zero. So *to solve $f(D)y = 0$ we can consider each factor separately* and if we can thus get n independent solutions, we can infer the general solution by what has been shown in § 13·1.

For example, the solution of $(D-m)y=0$ is evidently ce^{mx}. Hence the solution of

$$(D-m_1)(D-m_2)\ldots(D-m_n)\,y = 0,$$

where the factors are all different, is

$$c_1e^{m_1x}+c_2e^{m_2x}+\ldots+c_ne^{m_nx},$$

as before.

13·4. Auxiliary equation having equal roots. If the auxiliary equation has two equal roots, say $m_1=m_2$, the solution of the differential equation $f(D)y=0$ obtained in § 13·2 reduces to

$$(c_1+c_2)e^{m_1x}+c_3e^{m_3x}+\ldots+c_ne^{m_nx}$$

which has only $n-1$ arbitrary constants, because c_1+c_2 is equivalent to one arbitrary constant only.

To obtain the general solution, we notice that in this case the algebraical equation $f(D)=0$ has two roots equal to m_1, so that if we can get the general solution of the differential equation

$$(D-m_1)^2y=0, \quad . \quad . \quad . \quad (1)$$

we can obtain the general solution of $f(D)y=0$.

Put $(D-m_1)y=v$ in (1). Then it reduces to

$$(D-m_1)v=0,$$

or

$$\frac{dv}{dx}=m_1v.$$

Separating the variables, and integrating,

$$\log v=c_0+m_1x, \text{ or } v=ce^{m_1x},$$

i.e.,

$$(D-m_1)y=ce^{m_1x}.$$

This is a linear differential equation (§ 10·5). The solution is

$$ye^{-m_1x}=C+c\int e^{m_1x-m_1x}\,dx=C+cx.$$

Hence the general solution of (1) is

$$y=(c_1+c_2x)e^{m_1x}.$$

Consequently the general solution of $f(D)y=0$ in this case is

$$(\mathbf{c}_1+\mathbf{c}_2\mathbf{x})\mathbf{e}^{m_1x}+c_3e^{m_3x}+\ldots+c_ne^{m_nx}.$$

If in any case r roots are equal to m_1, the general solution of $f(D)y=0$ will be

$$(c_1+c_2x+c_3x^2+\ldots+c_rx^{r-1})e^{m_1x}+c_{r+1}e^{m_{r+1}x}$$
$$+\ldots+c_ne^{m_nx},$$

as is easy to see by putting first $(D-m_1)^{r-1}y$ equal to v_1 and solving for v_1, then putting $(D-m_1)^{r-2}y=v_2$ and solving for v_2, and so on.

Ex. Solve $\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}+y=0.$

Here $(D^3-D^2-D+1)y=0$, or $(D-1)^2(D+1)y=0$.

Hence the solution is $y=(c_1+c_2x)e^x+c_3e^{-x}$.

13·5. Auxiliary equation having imaginary roots. We suppose the coefficients $a_1, a_2, \ldots$ in $f(D)y=0$ to be all real. Hence if some roots of the algebraic equation $f(m)=0$ are imaginary, they will occur in pairs. Let $\alpha\pm i\beta$ be a pair of imaginary roots. Then the corresponding solution is

$$c_1e^{(\alpha+i\beta)x}+c_2e^{(\alpha-i\beta)x}.$$

We can transform this expression into a more convenient form. We have

$$\begin{aligned}c_1e^{(\alpha+i\beta)x}+c_2e^{(\alpha-i\beta)x}&=c_1e^{\alpha x}(\cos\beta x+i\sin\beta x)\\&\quad+c_2e^{\alpha x}(\cos\beta x-i\sin\beta x)\\&=e^{\alpha x}\{(c_1+c_2)\cos\beta x+(ic_1-ic_2)\sin\beta x\}\\&=e^{\alpha x}(A_1\cos\beta x+A_2\sin\beta x)\\&=\mathbf{C}_1\mathbf{e}^{\alpha x}\cos(\beta\mathbf{x}+\mathbf{C}_2),\end{aligned}$$

by an obvious change in the arbitrary constants.

Hence the general solution of $f(D)y=0$ when $f(m)=0$ has a pair of imaginary roots $\alpha \pm i\beta$ is

$$c_1 e^{\alpha x} \cos(\beta x + c_2) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}.$$

If there are two pairs of imaginary roots, $\alpha \pm i\beta$, $\gamma \pm i\epsilon$, the general solution of $f(D)y=0$ is

$$c_1 e^{\alpha x} \cos(\beta x + c_2) + c_3 e^{\gamma x} \cos(\epsilon x + c_4) + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x};$$

and so on.

If, however, $f(m)=0$ has two equal pairs of imaginary roots, say $\alpha + i\beta$ and $\alpha - i\beta$ occur twice, the general solution may be written as

$$e^{\alpha x}\{(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x\} + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x},$$

as can be easily seen by simplifying

$$(c_1 + c_2 x) e^{(\alpha + i\beta)x} + (c_3 + c_4) e^{(\alpha - i\beta)x};$$ and so on.

NOTE 1. $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ can also be written as

$$C_1 e^{\alpha x} \sin(\beta x + C_2)$$

by an obvious change in the arbitrary constants.

NOTE 2. If a pair of the roots of the auxiliary equation involves surds, say it is $\alpha \pm \sqrt{\beta}$, where β is positive, then the corresponding term in the C.F. can similarly be written as

$$C_1 e^{\alpha x} \cosh(x\sqrt{\beta} + C_2), \text{ or } C_1 e^{\alpha x} \sinh(x\sqrt{\beta} + C_2).$$

Ex. 1. Solve $\qquad (D^2+1)(D-1)y=0.$

The solution is $\qquad y = c_1 \cos x + c_2 \sin x + c_3 e^x.$

Ex. 2. Solve $\qquad (D^2+D+1)^2(D-2)y=0.$

Since $m^2+m+1=0$ has the roots $m=\frac{1}{2}\{-\pm i\sqrt{3}\}$, the solution of the given differential equation is

$$y = e^{-x/2}\{(c_1 + c_2 x) \cos(\tfrac{1}{2}x\sqrt{3}) + (c_3 + c_4 x) \sin(\tfrac{1}{2}x\sqrt{3})\} + c_5 e^{2x}.$$

EXAMPLES

Solve

1. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=0.$

2. $\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+9y=0.$

3. $\dfrac{d^2y}{dx^2}+2p\dfrac{dy}{dx}+(p^2+q^2)y=0.$

4. $\dfrac{d^3y}{dx^3}-2\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}-8y=0.$

5. $\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-2y=0.$

6. $\dfrac{d^4y}{dx^4}-2\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=0.$

7. $(D^4+8D^2+16)y=0.$

8. $\dfrac{d^4y}{dx^4}+m^4y=0.$ [*Agra*, 1954]

9. $\dfrac{d^2y}{dx^2}\pm\mu^2y=0.$

10. $\dfrac{d^2y}{dx^2}+y=0$, given $y=2$ for $x=0$, $y=-2$ for $x=\frac{1}{2}\pi$.

13·6. The particular integral. Let

$$\frac{1}{f(D)}Q \qquad \cdot \quad \cdot \quad \cdot \quad (1)$$

denote some function of x which when operated upon by $f(D)$ gives Q. Then this function of x is evidently a particular solution of the differential equation

$$f(D)y=Q, \qquad \cdot \quad \cdot \quad \cdot \quad (2)$$

for the substitution in (2) of (1) for y will satisfy the equation. We shall generally write $\{f(D)\}^{-1} Q$ for $\dfrac{1}{f(D)} Q$ for the sake of convenience.

Now the above definition of $\{f(D)\}^{-1} Q$ shows that

$$f(D)[\{f(D)\}^{-1}Q]=Q,$$

i.e., $f(D)$ and $\{f(D)\}^{-1}$ are inverse operations. In particular

$$D\{D^{-1}Q\}=Q, \text{ i.e., } \frac{d}{dx}\{D^{-1}Q\}=Q,$$

so that
$$\mathbf{D^{-1}Q=\int Q\,dx.}$$

However, in finding $\int Q\,dx$, we need not add any arbitrary constant, for we want only a particular integral. In fact we shall find in every case that the part in $\{f(D)\}^{-1} Q$ which involves an arbitrary constant is already included in the complementary function and its inclusion in the particular integral also will not make the solution different or more general. We may take for $\{f(D)\}^{-1} Q$ the simplest function of x which when operated upon by $f(D)$ will give Q.

The definition of $\{f(D)\}^{-1} Q$ shows that if

$$f(D)v=Q, \qquad \ldots \quad (1)$$

then
$$v=\{f(D)\}^{-1} Q. \qquad \ldots \quad (2)$$

To the student it may appear that we infer this by dividing by $f(D)$. But this is wrong, because $f(D)$ is not a number; it is an operator, i.e., it indicates that certain operations like differentiation, multiplication, addition, etc., are to be performed in a certain order on the quantity which follows it. We know that (2) is true, because we

see that if we operate on both sides with $f(D)$, the left-hand side gives $f(D)v$ and the right-hand side gives, by definition, Q, and these are equal by (1).

We have seen above that if the coefficients in $f(D)$ are constants, as we suppose to be the case, we can subject $f(D)$ in $f(D)y$ to any of the processes of algebra, and, in particular, factorise, $f(D)$ and take the factors in any order. It follows from the definition of $\{f(D)\}^{-1}Q$ that we can subject the $f(D)$ in

$$\frac{1}{f(D)}Q$$

also to any of the algebraic processes, and, in particular, factorise $f(D)$ and arrange the factors in any order, without affecting the value of the expression.

We shall now find the value of $(D-a)^{-1}Q$.

Put $$(D-a)^{-1}Q=v.$$

Then, by definition,

$$(D-a)v=Q,$$

i.e., $$\frac{dv}{dx}-av=Q,$$

which is a linear differential equation (§ 10·5). The solution is

$$v=ce^{\alpha x}+e^{\alpha x}\int e^{-\alpha x}Q\,dx.$$

Now c can be taken to be zero, for we want only a particular solution. Hence we may take that

$$\mathbf{(D-\alpha)^{-1}Q=e^{\alpha x}\int Qe^{-\alpha x}\,dx.}$$

We are now in a position to evaluate $\{f(D)\}^{-1}Q$. Let, on factorisation

$$f(D) \equiv (D-\alpha_1)(D-\alpha_2)\dots(D-\alpha_n).$$

Then, since

$$(D-\alpha_1)(D-\alpha_2)(D-\alpha_3)\dots(D-\alpha_n)y = Q,$$

it follows from the definition, that

$$\begin{aligned}(D-\alpha_2)(D-\alpha_3)\dots(D-\alpha_n)y \\ &= (D-\alpha_1)^{-1}Q \\ &= e^{\alpha_1 x}\int e^{-\alpha_1 x}\, Q\, dx.\end{aligned}$$

Therefore $(D-\alpha_3)\dots(D-\alpha_n)y$

$$\begin{aligned} &= (D-\alpha_2)^{-1}\left\{e^{\alpha_1 x}\int e^{-\alpha_1 x} Q\, dx\right\} \\ &= e^{\alpha_2 x}\int e^{(\alpha_1-\alpha_2)x}\int e^{-\alpha_1 x}\, Q\, dx\, dx;\end{aligned}$$

and so on.

Hence we shall get finally

$$y = e^{\alpha_n x}\int e^{(\alpha_{n-1}-\alpha_n)x}\int\dots\int e^{(\alpha_1-\alpha_2)x}\int e^{-\alpha_1 x}\, Q\, dx\; dx\dots dx.$$

This is the required particular integral.

This method is applicable even when the factors of $f(D)$ are not all different.

The alternative method of finding the particular integral given immediately below is generally easier. There are, moreover, special methods for finding the particular integral for special forms of Q which are easier still. These also are considered below.

13·7. The particular integral. Alternative method. Let $\{f(D)\}^{-1}$, regarded as an algebraical function of D be resolved into partial fractions; say

$$\frac{1}{f(D)} = \frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \dots + \frac{A_n}{D-\alpha_n}. \qquad (1)$$

It is assumed that a_1, a_2, ... are all different, so that there is no quadratic denominator. It will be shown now that

$$\frac{1}{f(D)}Q=\left\{\frac{A_1}{D-a_1}+\frac{A_2}{D-a_2}+\dots+\frac{A_n}{D-a_n}\right\}Q, \quad (2)$$

where the right-hand side means

$$A_1\cdot\frac{1}{D-a_1}Q+A_2\cdot\frac{1}{D-a_2}Q+\dots.$$

To prove (2), we must show that the result of operating on the right-hand side by $f(D)$ gives us Q.

Consider first the first term. We have

$$f(D)\left\{A_1\cdot\frac{1}{D-a_1}Q\right\}=A_1f(D)\left\{\frac{1}{D-a_1}Q\right\},$$

since A_1 is merely a constant,

$$=A_1(D-a_1)(D-a_2)\dots(D-a_n)\left\{\frac{1}{D-a_1}Q\right\},$$

by § 13·3,

$$=A_1(D-a_2)\dots(D-a_n)(D-a_1)\left\{\frac{1}{D-a_1}Q\right\},$$

by § 13·3,

$$=A_1(D-a_2)(D-a_3)\dots(D-a_n)\,Q, \text{ by § 13·6.}$$

Next consider the second term. We have, as in the case of the first term,

$$f(D\left\{A_2\cdot\frac{1}{(D-a_2)}Q\right\}$$
$$=A_2(D-a_1)(D-a_3)\dots(D-a_n)Q;$$

and so on.

Adding up these results, we see that

$$f(D)\left\{\Sigma \frac{A_1}{D-\alpha_1}\right\}Q$$
$$=\{\Sigma A_1(D-\alpha_2)(D-\alpha_3)\ldots(D-\alpha_n)\}Q. \qquad (3)$$

Now, by § 13·3, we can simplify $\Sigma A_1(D-\alpha_2)\ldots(D-\alpha_n)$ by treating D as an algebraic symbol, since $\alpha_1, \alpha_2, \ldots$ and $A_1, A_2, \ldots$ are constants. After simplification we shall find that

$$\Sigma A_1(D-\alpha_2)(D-\alpha_3)\ldots(D-\alpha_n)$$

is equal to 1, because this is the numerator when the partial fractions on the right-hand side in (1) are brought to a common denominator.

Hence (3) becomes

$$f(D)\left\{\Sigma \frac{A_1}{D-\alpha_1}\right\}Q=Q,$$

showing that (2) is valid.

This is the method which has to be employed when none of the special methods which follow is applicable. If $f(D)$ has imaginary factors, the terms in the particular integral corresponding to these should be combined into a real result. If $f(D)$ has repeated factors, some of the partial fractions will have non-linear denominators. The corresponding parts of the particular integral should then be found by applying § 13·6.

Ex. Solve $\dfrac{d^2y}{dx^2}+a^2y=\sec ax.$ [*Allahabad*, 1960]

The auxiliary equation is $m^2+a^2=0$, or $m=\pm ia$.

Hence the C.F. is $c_1 \cos ax+c_2 \sin ax$.

The P.I. $=\dfrac{1}{D^2+a^2}\sec ax=\dfrac{1}{(D+ia)(D-ia)}\sec ax$

$$=\frac{1}{2ia}\left\{\frac{1}{D-ia}-\frac{1}{D+ia}\right\}\sec ax.$$

Now

$$\frac{1}{D-ia}\sec ax = e^{iax}\int \frac{e^{-iax}}{\cos ax}\,dx = e^{iax}\int \frac{\cos ax - i\sin ax}{\cos ax}\,dx$$
$$= e^{iax}\{x + i(1/a)\log\cos ax\}.$$

Similarly $\dfrac{1}{D+ia}\sec ax = e^{-iax}\{x - i(1/a)\log\cos ax\}.$

Subtracting and dividing by $2ia$, we find that the P.I.

$$= \frac{1}{a}\left\{x\sin ax + \frac{1}{a}(\log\cos ax)\cos ax\right\}.$$

Hence the general solution is

$$y = c_1\cos ax + c_2\sin ax + (x\sin ax)/a + (\cos ax\log\cos ax)/a^2.$$

13·8. Special methods. $\{f(D)\}^{-1}e^{ax}$. We notice that

$$D^n e^{ax} = a^n e^{ax},\ D^{n-1}e^{ax} = a^{n-1}e^{ax},\ \ldots,$$
$$De^{ax} = ae^{ax},\ e^{ax} = e^{ax}.$$

Multiplying these equations by $1, a_1, a_2, \ldots, a_n$ respectively and adding,

$$f(D)e^{ax} = f(a)e^{ax}. \quad . \quad . \quad . \quad (1)$$

This suggests that probably

$$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax},$$

provided $f(a)\neq 0$. That the proposition is really true we can easily verify. For this it is only necessary to show that the result of operating on the right-hand side with $f(D)$ will give us e^{ax}.

Now $\quad f(D)\left\{\dfrac{1}{f(a)}e^{ax}\right\} = \dfrac{1}{f(a)}f(D)e^{ax}$, because $1/f(a)$ is a constant,

$$=\frac{1}{f(a)}f(a)e^{ax}, \text{ by (1)}, =e^{ax},$$

which proves the proposition.

If $f(a)=0$, we can apply § 13·84.

Ex. Solve $\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}+6y=e^{2x}$.

The auxiliary equation is

$$m^2+5m+6=0 \text{ or } (m+2)(m+3)=0.$$

Hence the solution is

$$y=c_1e^{-2x}+c_2e^{-3x}+\frac{1}{2^2+5.2+6}e^{2x},$$

or $$y=c_1e^{-2x}+c_2e^{-3x}+\tfrac{1}{20}e^{2x}.$$

EXAMPLES

Solve

1. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=e^{-x}$. [*Andhra*, 1960]

2. $\dfrac{d^2y}{dx^2}+2p\dfrac{dy}{dx}+(p^2+q^2)y=e^{ax}$.

3. $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y=e^{5x}$. [*Sagar*, 1954]

4. $\dfrac{d^2y}{dx^2}+31\dfrac{dy}{dx}+240y=272e^{-x}$. [*Banaras, Geophysics*, 1957]

5. $\dfrac{d^2y}{dx^2}-2k\dfrac{dy}{dx}+k^2y=e^x$. [*Nagpur*, 1957]

6. Obtain the complete solution of the differential equation

$$\frac{d^2y}{dx^2}-7\frac{dy}{dx}+6y=e^{2x},$$

and determine the constants so that $y=0$ when $x=0$. [*Lucknow*, 1955]

13·81. $\{\mathbf{f(D)}\}^{-1}$ sin ax and $\{(\mathbf{fD})\}^{-1}$ cos ax.

We notice that $D^2 \sin ax = -a^2 \sin ax$,

$$D^4 \sin ax = (-a^2)^2 \sin ax, \ldots,$$

$$D^{2n} \sin ax = (-a^2)^n \sin ax.$$

Hence $\quad f(D^2) \sin ax = f(-a^2) \sin ax. \qquad (1)$

This suggests that probably

$$\frac{1}{\mathbf{f(D^2)}} \mathbf{\sin ax} = \frac{1}{\mathbf{f(-a^2)}} \mathbf{\sin ax}, \qquad (2)$$

provided $f(-a^2) \neq 0$.

To see if this is true, we operate on the right-hand side with $f(D^2)$. We get

$$f(D^2)\left\{\frac{1}{f(-a^2)} \sin ax\right\} = \frac{1}{f(-a^2)}\{f(D^2) \sin ax\},$$

since $1/f(-a^2)$ is a constant,

$$= \frac{1}{f(-a^2)}\{f(-a^2) \, . \sin ax\}, \text{ by (1)},$$

$$= \sin ax.$$

Hence (2) is true.

If $f(D)$ contains odd powers of D also, it can be put in the form $f_1(D^2) + Df_2(D^2)$.

Then, $\quad \dfrac{1}{f(D)} \sin ax = \dfrac{1}{f_1(D^2) + Df_2(D^2)} \sin ax$

$$= \frac{1}{f_1(-a^2) + Df_2(-a^2)} \sin ax, \text{ by § 13·6 and eqn. (2) above,}$$

$$= \frac{1}{p + qD} \sin ax, \text{ say,}$$

$$=(p-qD)\left\{\frac{1}{p-qD}\left(\frac{1}{p+qD}\sin ax\right)\right\}, \text{ since } p-qD$$

and $1/(p-qD)$ are inverse operations,

$$=(p-qD)\left\{\frac{1}{p^2-q^2D^2}\sin ax\right\}$$

$$=(p-qD)\left\{\frac{1}{p^2+q^2a^2}\sin ax\right\},$$

the value of which can be found easily by differentiation. The case of $\cos ax$ can be treated similarly. We shall find that

$$\mathbf{\frac{1}{f(D^2)}\cos ax=\frac{1}{f(-a^2)}\cos ax.}$$

Ex. Solve $(D^2-3D+2)y=\sin 3x$. [*Nagpur*, 1949]

Since $D^2-3D+2=(D-1)(D-2)$,

so the C.F. $=c_1e^x+c_2e^{2x}$.

$$\text{The P.I.}=\frac{1}{D^2-3D+2}\sin 3x=\frac{1}{-9-3D+2}\sin 3x$$

$$=-\frac{1}{3D+7}\sin 3x=-(3D-7)\frac{1}{3^2D^2-7^2}\sin 3x$$

$$=-(3D-7)\frac{1}{3^2(-3^2)-7^2}\sin 3x$$

$$=\tfrac{1}{130}(3D-7)\sin 3x=\tfrac{1}{130}(9\cos 3x-7\sin 3x).$$

Hence the general solution is

$$y=c_1e^x+c_2e^{2x}+\tfrac{1}{130}(9\cos 3x-7\sin 3x).$$

13·82. $\{f(D)\}^{-1}\sin ax$, exceptional case. If D^2+a^2 is a factor of $f(D)$, the substitution of $-a^2$ for D^2 as required by the above method will make the denominator of the particular integral zero. Hence the above method fails in such cases. We have to apply § 13·7.

Thus, if $f(D)=(D^2+a^2)\,\phi(D)$, to evaluate $\{f(D)\}^{-1}\sin ax$ we can determine $(D^2+a^2)^{-1}\sin ax$ first and then apply the operator $\{\phi(D)\}^{-1}$ to the result.

Now $$\frac{1}{D^2+a^2}\sin ax=\frac{1}{(D+ia)(D-ia)}\sin ax$$
$$=\frac{1}{2ia}\left\{\frac{1}{D-ia}-\frac{1}{D+ia}\right\}\sin ax.$$

But
$$\frac{1}{D-ia}\sin ax=e^{iax}\int e^{-iax}\sin ax\,dx=e^{iax}\int\frac{e^{-iax}(e^{iax}-e^{-iax})}{2i}\,dx$$
$$=\frac{e^{iax}}{2i}\int(1-e^{-2iax})\,dx=\frac{e^{iax}}{2i}\left\{x+\frac{e^{-2iax}}{2ia}\right\}.$$

Similarly $$\frac{1}{D+ia}\sin ax=-\frac{e^{-iax}}{2i}\left\{x-\frac{e^{2iax}}{2ia}\right\}.$$

Hence $$\frac{1}{D^2+a^2}\sin ax=\frac{1}{2ia}\left\{\frac{\cos ax}{i}x-\frac{\sin ax}{2ia}\right\}$$
$$=-\frac{x\cos ax}{2a}+\frac{\sin ax}{4a^2}.$$

Also, if the differential equation is $(D^2+a^2)y=\sin ax$, the complementary function is $c_1\sin ax+c_2\cos ax$, showing that the term $(1/4a^2)\sin ax$ in the P. I. found above can be absorbed in the term $c_1\sin ax$ of the C.F. Hence we may write simply

$$\mathbf{\frac{1}{D^2+a^2}\sin ax=-\frac{x}{2a}\cos ax.}$$

We can show similarly that

$$\mathbf{\frac{1}{D^2+a^2}\cos ax=\frac{x}{2a}\sin ax.}$$

[A little consideration will show that the terms omitted above can be omitted even when the differential equation is $\phi(D)(D^2+a^2)y=\sin ax$, for $\{1/\phi(D)\}\sin ax$ will give rise only to terms of the type $c\cos ax$ or $c\sin ax$, where c is some constant, and both these are absorbable in the C.F.]

Ex. Solve $$\frac{d^3y}{dx^3}+a^2\frac{dy}{dx}=\sin ax.$$

The C.F. is $c_1+c_2\sin ax+c_3\cos ax$.

The P.I. is $\frac{1}{D}\cdot\frac{1}{D^2+a^2}\sin ax=\frac{1}{D}\left(\frac{-x\cos ax}{2a}\right)$ by the above,

$$=-\frac{x\sin ax}{2a^2}-\frac{\cos ax}{2a^3}.$$

Hence the complete solution is

$$y=c_1+c_2\sin ax+c_3\cos ax-(x\sin ax)/2a^2,$$

the last term in the P.I. being omitted, as it can be absorbed in the C.F.

Examples

Solve

1. $\frac{d^2y}{dx^2}+y=\cos 2x.$ [*Utkal*, 1956]

2. $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\sin 2x.$ [*Aligarh*, 1960]

3. $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+9y=40\sin 5x.$ [*Bombay*, 1947]

4. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=\sin 3x.$ [*P.S.C., U.P.*, 1957]

5. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=\cos 3x.$ [*Nagpur*, 1953]

6. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+y=a\sin 2x.$ [*Nagpur*, 1956]

7. Find the integral of the equation

$$\frac{d^2x}{dt^2}+2n\cos\alpha\frac{dx}{dt}+n^2x=a\cos nt,$$

which is such that when $t=0$, $x=0$ and $dx/dt=0$. [*Delhi*, 1960]

Solve

8. $\dfrac{d^2y}{dx^2}+9y=\cos 2x+\sin 2x.$

9. $\dfrac{d^2y}{dx^2}+4y=e^x+\sin 2x.$ [*Sagar*, 1960]

10. $\dfrac{d^2y}{dx^2}-4y=e^x+\sin 2x.$ [*Jammu*, 1951]

13·83. $\{f(D)\}^{-1}x^m$, where m is a positive integer. Consider first $(D-a)^{-1}x^m$. By § 13·6,

$$\frac{1}{D-a}x^m=e^{ax}\int e^{-ax}x^m\,dx$$

$$=e^{ax}\left\{-\frac{x^m e^{-ax}}{a}-\frac{mx^{m-1}e^{-ax}}{a^2}-\frac{m(m-1)x^{m-2}e^{-ax}}{a^3}\right.$$
$$\left.-\dots-\frac{m(m-1)\dots 2.1.e^{-ax}}{a^{m+1}}\right\}, \qquad (1)$$

by repeatedly integrating by parts.

But if we expand $1/(D-a)$ in powers of D, we get

$$\frac{1}{D-a}x^m=-\frac{1}{a(1-D/a)}x^m$$

$$=-\frac{1}{a}\left(1+\frac{D}{a}+\frac{D^2}{a^2}+\frac{D^3}{a^3}+\dots\right)x^m$$

$$=-\frac{1}{a}\left\{x^m+\frac{mx^{m-1}}{a}+\frac{m(m-1)x^{m-2}}{a^2}+\dots\right.$$
$$\left.+\frac{m(m-1)\dots 2.1}{a^m}\right\},$$

which is the same as (1).

Now we have shown in § 13·7 that we can break up

$$\frac{1}{f(D)} \text{ in } \frac{1}{f(D)}Q$$

into its partial fractions and consider each partial fraction separately, and we have shown above that we can expand each partial fraction in powers of D. We know also that the expansion of $\{f(D)\}^{-1}$ in powers of D will be the same whether we first break it into its partial fractions and then expand each term, or we expand it otherwise. Hence to evaluate $\{f(D)\}^{-1}x^m$ we can expand $\{f(D)\}^{-1}$ in powers of D by any method we like and operate upon x^m with the expansion obtained. As all the differential coefficients of x^m of an order higher than m are zero, the particular integral will consist of a finite number of terms only.

Ex. Solve $\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}=1+x^2.$

Since $\qquad D^3-D^2-6D=D(D+2)(D-3),$

the C.F. $=c_1+c_2e^{-2x}+c_3e^{3x}.$

$$\begin{aligned}\text{The P.I.} &= \frac{1}{D(D^2-D-6)}(1+x^2)\\ &= -\tfrac{1}{6}D^{-1}(1+\tfrac{1}{6}D-\tfrac{1}{6}D^2)^{-1}(1+x^2)\\ &= -\tfrac{1}{6}D^{-1}(1-\tfrac{1}{6}D+\tfrac{1}{6}D^2+\tfrac{1}{36}D^2)(1+x^2)\\ &= -\tfrac{1}{6}(D^{-1}-\tfrac{1}{6}+\tfrac{7}{36}D+\dots)(1+x^2)\\ &= -\tfrac{1}{6}(x+\tfrac{1}{3}x^3-\tfrac{1}{6}x^2+\tfrac{7}{18}x)+\text{a constant, which can}\end{aligned}$$

be omitted since the C.F. contains an arbitrary constant.

Hence the complete solution is

$$y=c_1+c_2e^{-2x}+c_3e^{3x}-\tfrac{1}{18}x^3+\tfrac{1}{36}x^2-\tfrac{25}{108}x.$$

EXAMPLES

Solve

1. $\dfrac{d^3y}{dx^3}-4\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}-2=0.$

2. $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y=x.$ [*Sagar*, 1954]

3. $y'''-y''-y'+y=x.$

4. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+2y=x.$ [*Nagpur*, 1952]

5. $\dfrac{d^2y}{dx^2}-4y=x^2.$ [*Gauhati*, '60]

6. $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=x^2.$

7. $\dfrac{d^3y}{dx^3}+3\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}=x^2.$ [*Aligarh*, 1958]

8. $\dfrac{d^3y}{dx^3}-3\dfrac{dy}{dx}-2y=x^2.$

9. $\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+8y=x.$

10. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-2y=x+\sin x.$ [*Agra*, 1959]

11. $\dfrac{d^2y}{dx^2}+4y=\sin^2 x.$

12. $\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=e^{2x}+x^2+x.$ [*Baroda*, 1959]

13·84. $\{\mathbf{f(D)}\}^{-1}\mathbf{e}^{ax}\,\mathbf{V}$, where V is any function of x. By successive differentiation we see that

$$D(e^{ax}V)=e^{ax}DV+ae^{ax}V=e^{ax}(D+a)V,$$
$$D^2(e^{ax}V)=e^{ax}(D+a)^2V,\ldots$$
$$D^n(e^{ax}V)=e^{ax}(D+a)^nV.$$

Therefore $f(D)(e^{ax}V)=e^{ax}f(D+a)V.$ . . . (1)

This suggests that probably

$$\frac{1}{\mathbf{f(D)}}(\mathbf{e}^{ax}\mathbf{V})=\mathbf{e}^{ax}\frac{1}{\mathbf{f(D+a)}}\mathbf{V}.$$

To test whether this is true, operate on the right-hand side with $f(D)$. We have

$$f(D)\left[e^{ax}\left\{\frac{1}{f(D+a)}V\right\}\right]$$

$$=e^{ax}f(D+a)\left\{\frac{1}{f(D+a)}V\right\}, \text{ by (1)},$$

$=e^{ax}V$, since $f(D+a)$ and $\{f(D+a)\}^{-1}$ are inverse operations, showing that the above proposition is true.

This proposition enables us to find $\{f(D)\}^{-1}e^{ax}$ when $f(a)$ is zero, as in Ex. 2 worked out below.

Occasionally it is more convenient to find the P.I. corresponding to $\cos ax$ or $\sin ax$ or to expressions involving these functions, by regarding $\cos ax$ or $i\sin ax$ as the real or the imaginary part of e^{iax} as in Ex. 4. worked out below.

Ex. 1. Solve $(D^2-2D+5)y=e^{2x}\sin x$. [*Baroda*, 1959]

The roots of the auxiliary equation are $1\pm\sqrt{(1-5)}$, i.e., $1\pm2i$. Hence the C.F. $=e^x(c_1\sin 2x+c_2\cos 2x)$.

The P.I. $=\dfrac{1}{D^2-2D+5}e^{2x}\sin x=e^{2x}\dfrac{1}{(D+2)^2-2(D+2)+5}\sin x$

$$=e^{2x}\frac{1}{D^2+2D+5}\sin x=e^{2x}\frac{1}{-1+2D+5}\sin x$$

$$=\tfrac{1}{2}e^{2x}\frac{1}{D+2}\sin x=\tfrac{1}{2}e^{2x}.(D-2)\frac{1}{D^2-4}\sin x$$

$$=\tfrac{1}{2}e^{2x}(-\tfrac{1}{5})(D-2)\sin x=-\tfrac{1}{10}e^{2x}(\cos x-2\sin x).$$

Hence the general solution is

$$y=e^x(c_1\sin 2x+c_2\cos 2x)-\tfrac{1}{10}e^{2x}(\cos x-2\sin x).$$

Ex. 2. Solve $(D^2+D-2)y=e^x$.

Since $D^2+D-2=(D+2)(D-1)$, the C.F. is

$$c_1e^x+c_2e^{-2x}.$$

By the method of § 13·8, the P.I. $=\dfrac{e^x}{1^2+1-2}=\dfrac{e^x}{0}$, which is meaningless. But if we regard e^x as the product of e^x and 1, and apply the method of the present article, we get

$$\frac{1}{(D-1)(D+2)}e^x=\frac{1}{(D-1)}\left\{\frac{1}{(D+2)}e^x\right\}=\frac{1}{D-1}\left(\tfrac{1}{3}e^x\right)$$

$$=\tfrac{1}{3}\cdot\frac{1}{D-1}(e^x\,.\,1)=\tfrac{1}{3}e^x\frac{1}{D+1-1}1=\tfrac{1}{3}e^x D^{-1}1=\tfrac{1}{3}xe^x.$$

Hence the general solution is $y=c_1e^x+c_2e^{-2x}+\frac{1}{3}xe^x$.

This method will succeed even if the factor which becomes zero occurs more than once in the denominator, as in the following example.

Ex. 3. Solve $(D+2)(D-1)^3y=e^x$.

The C.F. is $(c_1+c_2x+c_3x^2)e^x+c_4e^{-2x}$.

The P.I. $=\dfrac{1}{(D-1)^3}\left\{\dfrac{1}{D+2}e^x\right\}=\dfrac{1}{3}\dfrac{1}{(D-1)^3}(e^x\,.\,1)=\dfrac{1}{3}e^x\dfrac{1}{D^3}1$

$$=\tfrac{1}{3}e^x\frac{1}{D^2}x=\tfrac{1}{3}e^x\frac{1}{D}\left(\tfrac{1}{2}x^2\right)=\tfrac{1}{18}x^3e^x.$$

Hence the general solution is

$$y=(c_1+c_2x+c_3x^2)e^x+c_4e^{-2x}+\tfrac{1}{18}x^3e^x.$$

Note. If in the last example we take the general value of $D^{-3}1$, which is $\frac{1}{6}x^3+ax^2+bx+c$, where a, b, c are arbitrary constants, or if we take any particular value of this other than $\frac{1}{6}x^3$, we would not get a more general solution of the differential equation, because the terms involving a, b, c are included in the complementary function.

Ex. 4. Solve $y''+a^2y=\sin ax$. [*Banaras*, 1956]

We have already solved this example in § 13·82. The following is a shorter method.

$\dfrac{1}{D^2+a^2}\sin ax=$coeff. of i in the value of

$$\frac{1}{D^2+a^2}(\cos ax+i\sin ax),\text{ i.e., of }\frac{1}{D^2+a^2}e^{iax}.$$

But $\dfrac{1}{D^2+a^2}e^{iax}=\dfrac{1}{D-ia}\left(\dfrac{1}{D+ia}e^{iax}\right)$

$$=\frac{1}{D-ia}\left(\frac{1}{2ia}e^{iax}\right), \text{ by § 13·8,}$$

$$=e^{iax}\frac{1}{D}\left(\frac{1}{2ia}\right), \text{ by § 13·84,}$$

$$=e^{iax}\frac{x}{2ia}=-\frac{ix}{2a}\left(\cos ax+i\sin ax\right).$$

Hence, equating the imaginary parts,

$$\frac{1}{D^2+a^2}\sin ax=-\frac{x}{2a}\cos ax,$$

so that the solution of the differential equation is

$$y=c_1\sin ax+c_2\cos ax-(x/2a)\cos ax.$$

[Note that the above shows at once, on equating the real parts, that $\dfrac{1}{D^2+a^2}\cos ax=\dfrac{x}{2a}\sin ax$.]

Examples

Solve

1. $(D^2-3D+2)y=xe^x$.
2. $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+y=e^{2x}\sin 2x$. [*Nagpur*, 1954]
3. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+4y=e^x\cos x$. [*P.S.C., U.P.*, 1956]
4. $\dfrac{d^4y}{dx^4}-y=e^x\cos x$. [*Aligarh*, 1960]
5. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=x^2e^{3x}$. [*Banaras*, 1960]
6. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=e^x$. [*Calcutta*, 1954]
7. $(D^2-3D+2)y=e^x$, if $y=3$ and $Dy=3$ when $x=0$.

8. $(D^2-1)y=\cosh x \cos x.$

(A.U.77)

9. $\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}-2y=e^x+\cos x.$ [*Allahabad*, 1954] —77.

10. $4\dfrac{d^2y}{dx^2}+12\dfrac{dy}{dx}+9y=144e^{-3x/2}.$

11. $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+4y=e^{2x}-e^{-2x}.$

12. $\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}-y=xe^x+e^x.$

13. $\dfrac{d^3y}{dx^3}-7\dfrac{dy}{dx}-6y=e^{2x}(1+x).$ [*Ujjain*, 1960]

14. $\dfrac{d^4y}{dx^4}+\dfrac{d^2y}{dx^2}+y=ax^2+be^{-x}\sin 2x.$ [*Delhi, Hons.*, 1960]

15. $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}-12y=(x-1)e^{2x}.$ [*P.S.C., U.P.*, 1955]

13·85. $\{\mathbf{f(D)}\}^{-1}\mathbf{xV}$, where V is any function of x. By Leibnitz's theorem

$$D^n(xV)=xD^nV+nD^{n-1}V$$

$$=xD^nV+\left(\frac{d}{dD}D^n\right)V,$$

showing that $f(D)xV=xf(D)V+f'(D)V.$ (1)

This suggests that probably

$$\frac{\mathbf{1}}{\mathbf{f(D)}}\mathbf{xV}=\mathbf{x}\frac{\mathbf{1}}{\mathbf{f(D)}}\mathbf{V}+\left\{\frac{\mathbf{d}}{\mathbf{dD}}\frac{\mathbf{1}}{\mathbf{f(D)}}\right\}\mathbf{V},$$

where $\dfrac{d}{dD}\dfrac{1}{f(D)}V$ is a symbolic method of writing

$$-f'(D)\left[\frac{1}{\{f(D)\}^2}\right]V.$$

We can easily see that this is true, for operating on the right-hand side by $f(D)$ we get

$$f(D)\left[x\frac{1}{f(D)}V+\left\{\frac{d}{dD}\frac{1}{f(D)}\right\}V\right]$$

$$=xf(D)\frac{1}{f(D)}V+f'(D)\frac{1}{f(D)}V-f(D)\left\{f'(D)\frac{1}{\{f(D)\}^2}\right\}V,$$

by (1),

$$=xV+f'(D)\frac{1}{f(D)}V-f'(D)\left[f(D)\left\{\frac{1}{f(D)}\left(\frac{1}{f(D)}V\right)\right\}\right]$$

$$=xV.$$

We can show similarly that

$$\frac{1}{f(D)}x^mV=x^m\frac{1}{f(D)}V+mx^{m-1}\left\{\frac{d}{dD}\frac{1}{f(D)}\right\}V$$
$$+\frac{m(m-1)}{2!}x^{m-2}\left\{\frac{d^2}{dD^2}\frac{1}{f(D)}\right\}V+\dots.$$

The above formula is not very convenient in practice. So in finding the particular integral corresponding to x^me^{ax}, § 13·84 should be applied. In evaluating the particular integral corresponding to $x^m \sin ax$ or $x^m \cos ax$, it is frequently more convenient to replace $\sin ax$ or $\cos ax$ by the exponential value and apply § 13·84, rather than apply the present formula.

Ex. 1. Solve $(D^2+2D+1)y=x\cos x$. [*Baroda*, 1959]

The C.F. is evidently $(c_1+c_2x)e^{-x}$.

$$\text{The P.I.}=\frac{1}{D^2+2D+1}x\cos x$$

$$=x\frac{1}{D^2+2D+1}\cos x-2(D+1)\frac{1}{(D^2+2D+1)^2}\cos x$$

$$=x\frac{1}{2D}\cos x-2(D+1)\frac{1}{4D^2}\cos x$$

$$=\tfrac{1}{2}x\sin x+\tfrac{1}{2}(D+1)\cos x=\tfrac{1}{2}x\sin x-\tfrac{1}{2}\sin x+\tfrac{1}{2}\cos x.$$

Hence the general solution is

$$y=(c_1+c_2x)e^{-x}+\tfrac{1}{2}(x-1)\sin x+\tfrac{1}{2}\cos x.$$

Ex. 2. Solve $\dfrac{d^4y}{dx^4}+2\dfrac{d^2y}{dx^2}+y=x^2\cos x.$ [*Vikram*, '61]

The auxiliary equation is $(m^2+1)^2=0$.

Hence the C.F. is $(c_1+c_2x)\sin x+(c_3+c_4x)\cos x$.

$$\text{The P.I.}=\frac{1}{(D^2+1)^2}\tfrac{1}{2}x^2(e^{ix}+e^{-ix}).$$

Now
$$\frac{1}{(D^2+1)^2}\tfrac{1}{2}x^2e^{ix}=\tfrac{1}{2}e^{ix}\frac{1}{\{(D+i)^2+1\}^2}x^2$$
$$=\tfrac{1}{2}e^{ix}\frac{1}{(D^2+2iD)^2}x^2$$
$$=\tfrac{1}{2}e^{ix}\frac{1}{-4D^2(1-\frac{1}{2}iD)^2}x^2$$
$$=-\tfrac{1}{8}e^{ix}\frac{1}{D^2}(1+iD-\tfrac{3}{4}D^2+\ldots)x^2$$
$$=-\tfrac{1}{8}e^{ix}(D^{-2}+iD^{-1}-\tfrac{3}{4}+\ldots)x^2$$
$$=-\tfrac{1}{8}e^{ix}(\tfrac{1}{12}x^4+\tfrac{1}{3}ix^3-\tfrac{3}{4}x^2)+\text{terms in } x^1,\ x^0.$$

Similarly
$$\frac{1}{(D^2+1)^2}\tfrac{1}{2}x^2e^{-ix}=-\tfrac{1}{8}e^{-ix}(\tfrac{1}{12}x^4-\tfrac{1}{3}ix^3-\tfrac{3}{4}x^2)+\text{terms in } x^1,\ x^0.$$

By addition, the P.I. required

$$=-\tfrac{1}{4}(\tfrac{1}{12}x^4\cos x-\tfrac{1}{3}x^3\sin x-\tfrac{3}{4}x^2\cos x)+\text{terms included in the C.F.}$$

Hence the general solution is

$$y=(c_1+c_2x)\sin x+(c_3+c_4x)\cos x+\tfrac{1}{12}x^3\sin x+\tfrac{1}{48}(9x^2-x^4)\cos x.$$

Alternative Method. Another method of finding the P.I. is to find the P.I. corresponding to x^2e^{ix} as above and take the real part. (Compare with the method used in § 13·84, Ex. 4.)

EXAMPLES

Solve

1. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=x\sin x.$ [*Aligarh*, 1946]

2. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=x\cos x.$ [*Ujjain*, 1960]

3. $\dfrac{d^2y}{dx^2}+y=x^2\sin 2x.$ 26 [*Aligarh*, 1945]

4. $(D^2+1)^2y=24x\cos x,$
given the initial conditions $x=0,\ y=0,\ Dy=0,\ D^2y=0,\ D^3y=12.$

5. $\dfrac{d^2y}{dx^2}-y=x\sin x+(1+x^2)e^x.$ [*Banaras*, 1958]

EXAMPLES ON CHAPTER XIII

1. $\dfrac{d^2y}{dx^2}+(a+b)\dfrac{dy}{dx}+aby=0.$

2. $\dfrac{d^2y}{dx^2}-2a\dfrac{dy}{dx}+a^2y=0.$

3. $(D^4+2D^3+3D^2+2D+1)y=0.$

4. $\dfrac{d^4y}{dx^4}+13\dfrac{d^2y}{dx^2}+36y=0.$

5. $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}-12y=e^{3x}.$ 76.

6. $\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}+6y=e^x.$ [*Nagpur*, 1953]

7. $3\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-14y=13e^{2x}.$ [*Madras*, 1950]

8. $(2D+1)^2y=4e^{-x/2}.$

9. $\dfrac{d^2y}{dx^2}-y=\cosh x.$ [*Ban.*, 54] 10. $\dfrac{d^2y}{dt^2}-3\dfrac{dy}{dt}+2y=10\sin t.$

11. $\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-2y=\sin 2x.$

12. $\dfrac{d^2y}{dx^2}+2k\dfrac{dy}{dx}+k^2y=A\cos px.$

13. $\dfrac{d^2y}{dx^2}+a^2y=\cos ax.$ [*Vikram*, 1961]

14. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+10y+37\sin 3x=0$; and find the value of y when $x=\frac{1}{2}\pi$ if it is given that $y=3$ and $dy/dx=0$ when $x=0$. [*Banaras*, 1956]

15. $\dfrac{d^2y}{dx^2}-13\dfrac{dy}{dx}+12y=x.$

16. $\dfrac{d^4y}{dx^4}+\dfrac{d^2y}{dx^2}+16y=16x^2+256.$

17. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+3y=\cos x+x^2.$ [*Panjab*, 1954]

18. $(D-1)^2(D^2+1)^2y=\sin^2\frac{1}{2}x+e^x.$ [*Gorakhpur*, 1960]

19. $\dfrac{d^4y}{dx^4}+\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-2y=x^2+e^x.$

20. $(D^5-D)y=12e^x+8\sin x-2x.$ [*Banaras*, 1957]

21. $(D^4+D^2+1)y=e^{-x/2}\cos(x\sqrt{3}/2).$ [*Gorakhpur*, 1960]

22. $\dfrac{d^3y}{dx^3}+3\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+y=e^{-x}.$ [*Panjab*, 1945]

23. $D^2y-3Dy+2y=\cosh x$ and verify the solution you obtain. [*First part*, *Allahabad*, 1948]

24. $\dfrac{d^2y}{dx^2}+y=e^{-x}+\cos x+x^3+e^x\sin x.$ [*P.S.C., U.P.*, 1960]

25. $\dfrac{d^4y}{dx^4}-y=x\sin x.$ [*Lucknow*, 1957]

26. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=xe^x\sin x.$ [*P.S.C., U.P.*, 1952]

27. $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=8x^2e^{2x}\sin 2x.$ [*Allahabad*, 1960]

28. Solve the equation

$$\frac{d^2y}{dx^2}=a+bx+cx^2,$$

given that $dy/dx=0$ when $x=0$ and $y=d$ when $x=0$.

[*Banaras*, 1959]

29. If
$$\frac{d^2x}{dt^2}+\frac{g}{b}(x-a)=0,$$
(a, b and g being positive numbers) and $x=a'$ and $dx/dt=0$ when $t=0$, show that

$$x=a+(a'-a)\cos\{\sqrt{(g/b)}\,t\}.$$ [*Banaras*, 1960]

30. Find the solution of the equation

$$\frac{d^2y}{dx^2}-y=1,$$

which vanishes when $x=0$ and tends to a finite limit as $x\to-\infty$.

[*Allahabad*, 1960]

CHAPTER XIV

MISCELLANEOUS DIFFERENTIAL EQUATIONS

14·1. Homogeneous linear equations. A differential equation of the form

$$x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}x\frac{dy}{dx}+a_ny=Q, \quad (1)$$

where a_1, a_2, ..., a_n are constant and Q is a function of x, is called a *homogeneous linear differential equation.*

1. By taking a new independent variable z, where

$$z=\log x,$$

the homogeneous linear equation becomes a linear equation with constant coëfficients. For

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz},$$

and
$$\frac{d^2y}{dx^2}=\frac{1}{x}\frac{d}{dz}\left(\frac{1}{x}\frac{dy}{dz}\right)=\frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right).$$

Similarly
$$\frac{d^3y}{dx^3}=\frac{1}{x^3}\left(\frac{d^3y}{dz^3}-3\frac{d^2y}{dz^2}+2\frac{dy}{dz}\right);\text{ etc.}$$

Writing D for d/dz, the above equations become

$$x(dy/dx)=Dy.$$
$$x^2(d^2y/dx^2)=D(D-1)y,$$
$$x^3(d^3y/dx^3)=D(D-1)(D-2)y;\text{ etc.}$$

We can, in fact, prove by mathematical induction that

$$x^n(d^ny/dx^n)=D(D-1)(D-2)\ldots(D-n+1)y. \qquad (2)$$

The substitution of these values of $x(dy/dx)$, $x^2(d^2y/dx^2)$, etc., will evidently give us a linear differential equation with constant coefficients, which can be solved by the methods of the last chapter.

2. There is, however, another method which is easier. Let θ denote the operator

$$x\frac{d}{dx},$$

so that
$$\theta y \text{ means } x\frac{dy}{dx},$$

$$\theta^2 y \text{ means } x\frac{d}{dx}\left(x\frac{dy}{dx}\right), \text{ etc.}$$

Now the symbol D used above stands for

$$\frac{d}{dz}, \text{i.e., } x\frac{d}{dx},$$

so that the operator θ is equivalent to the operator D, the only difference being that θ can be applied to functions of x directly, but D can be applied only after transforming them into functions of z. Hence equation (2) can be written as

$$\mathbf{x}^n \frac{\mathbf{d}^n\mathbf{y}}{\mathbf{dx}^n} = \theta(\theta-\mathbf{1})(\theta-\mathbf{2})\ldots(\theta-\mathbf{n}+\mathbf{1})\mathbf{y}.$$

Let now the differential equation (1) be transformed by this relation, so that we get, say, the equation

$$f(\theta)y = Q. \quad . \quad . \quad . \quad (3)$$

It is easy to show, as in the case of the linear equation with constant coefficients, that the general solution of (3) is the sum of any particular solution of (3) and the general solution of

$$f(\theta)y = 0. \quad . \quad . \quad . \quad (4)$$

To solve this last equation, assume tentatively that $y = x^m$. Substitution gives at once the result that x^m is a solution of (4) if m is a root of the *auxiliary equation*

$$f(m) = 0. \quad . \quad . \quad . \quad (5)$$

If $m_1, m_2, \ldots, m_n$ are the roots of (5), and no two of them are equal, the general solution of (4) is easily seen to be

$$y = c_1x^{m_1} + c_2x^{m_2} + c_3x^{m_3} + \ldots + c_nx^{m_n}.$$

This is the *complementary function* in the solution of equation (3).

3. The *particular integral* of (3), written as

$$\frac{1}{f(\theta)}Q, \qquad . \quad . \quad . \quad (6)$$

is some function of x which when operated upon by $f(\theta)$ gives Q.

It can be shown, just as in the case of linear equations with constant coefficients, that the operator θ satisfies the laws of algebra, and may be treated like an algebraical quantity in $f(\theta)$ provided the coefficients are constants. Further, we can prove with the help of this property that if $f(\theta)$ can be broken into factors $\theta-a_1$, $\theta-a_2, \ldots,$ $\theta-a_n$, say, then (6) will be equal to

$$\frac{1}{\theta-a_1}\left[\frac{1}{\theta-a_2}\left\{\frac{1}{\theta-a_3}\left(\cdot\cdot\left(\frac{1}{\theta-a_n}Q\right)\right)\right\}\right]. \qquad (7)$$

4. As an alternative, $\{f(\theta)\}^{-1}$ may be broken up into partial fractions, say

$$A_1/(\theta-a_1),\ A_2/(\theta-a_2),\ \ldots,\ A_n/(\theta-a_n);$$

then it can be shown, exactly as in the last chapter, that $\{f(\theta)\}^{-1}Q$ is equal to

$$\left\{\frac{A_1}{\theta-a_1}+\frac{A_2}{\theta-a_2}+\ldots+\frac{A_n}{\theta-a_n}\right\}Q. \qquad (8)$$

5. Now if $\dfrac{1}{\theta-a}Q=v$, then $(\theta-a)v=Q$,

i.e., $$x\frac{dv}{dx}-av=Q, \text{ or } \frac{dv}{dx}-\frac{a}{x}v=\frac{Q}{x},$$

which is a linear differential equation of the first order (§ 10·5).

The integrating factor is $e^{-a\log x}$, i.e., x^{-a}, and the solution is

$$\mathbf{v}=\mathbf{x}^{a}\int \mathbf{Q}\mathbf{x}^{-a-1}\,\mathbf{dx}. \qquad . \quad . \quad . \quad (9)$$

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This enables us to write down the values of (7) and (8) in terms of integrals.

6. In case Q is some power of x, say x^m, we may guess that

$$\frac{1}{f(\theta)}x^m=\frac{1}{f(m)}x^m, \quad . \quad . \quad . \quad (10)$$

since $f(\theta)x^m=f(m)x^m$. By operating on the right-hand side with $f(\theta)$ we can see at once that formula (10) is correct. This furnishes us with a short method of finding the particular integral in such cases.

7. In case the auxiliary equation (5) has r roots each equal to m, the corresponding part of the solution of (4) can be inferred from what we know about the solution of the linear equation with constant coefficients which results on using the substitution (2), and is equal to

$$x^{m_1}\{c_1+c_2\log x+\ldots+c_r(\log x)^{r-1}\}.$$

8. Similarly, if two roots of (5) are imaginary and equal to $\alpha\pm i\beta$, the corresponding part of the solution is

$$x^\alpha\{c_1\cos(\beta\log x)+c_2\sin(\beta\log x)\}.$$

Ex. Solve $(x^2D^2-3xD+4)y=2x^2$, where $D\equiv d/dx$.
[*Utkal*, 1949]

Since* $xD=\theta$, and $x^2D^2=\theta(\theta-1)$, the differential equation can be written as $\{\theta(\theta-1)-3\theta+4\}y=2x^2$,

or $$(\theta^2-4\theta+4)y=2x^2.$$

*The student should carefully note that the equation $xD=\theta$ means merely that *the operator xD is equivalent to the operator θ*; or in other words, if y is any function of x, that $xDy=\theta y$.

The auxiliary equation is $m^2-4m+4=0$, which has two roots each equal to 2. Hence the complementary function is

$$x^2(c_1+c_2\log x).$$

The P.I. $=\dfrac{1}{(\theta-2)^2}\,2x^2=2\,\dfrac{1}{(\theta-2)^2}\,x^2.$

The short method given above (formula 10) fails, for $\theta-2$ becomes zero when θ is put equal to 2. The method of breaking up $\{f(\theta)\}^{-1}$ into partial fractions also fails, because there is a repeated root. But formula (9) gives

$$\frac{1}{\theta-2}\,x^2=x^2\int x^2x^{-3}\,dx=x^2\log x.$$

Therefore, applying formula (9) once more,

$$\begin{aligned}\frac{1}{(\theta-2)^2}x^2&=\frac{1}{\theta-2}\,(x^2\log x)\\&=x^2\int x^{-3}\,x^2\log x\,dx\\&=\tfrac{1}{2}x^2\,(\log x)^2.\end{aligned}$$

Hence the general solution of the given differential equation is

$$y=x^2(c_1+c_2\log x)+(x\log x)^2.$$

Examples

Solve

1. $x^2D^2y+5x\,Dy+4y=0.$
2. $x^2y_2+xy_1=a.$
3. $(x^4D^3+2x^3D^2-x^2D+x)y=1.$
4. $x^2\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+6y=x.$ [*Baroda*, 1956]
5. $x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}=6x.$ [*Rajasthan*, 1962]
6. $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-4y=x^2.$ [*Banaras*, 1951]
7. $x^3\dfrac{d^3y}{dx^3}+2x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}-3y=x^2+x.$ [*Osmania*, '60]

8. $(x^2D^2+xD-1)y=x^3$. [*Bombay*, 1936]

9. $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=x^{-1}$. [*Nagpur*, 1954]

10. $(x^2D^2+2xD)\,y=\log x$.

11. $x^2\dfrac{d^2y}{dx^2}+7x\dfrac{dy}{dx}+13y=\log x$. [*Delhi*, 1958]

12. $x^3\dfrac{d^3y}{dx^3}+3x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=x+\log x$. [*Ban. Geoph.*, '60]

13. $(x^2D^2-xD+2)\,y=x\log x$. [*Delhi*, 1960]

14. $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-3y=x^2\log_e x$. [*Baroda*, 1959]

14·2. Simultaneous linear differential equations with constant coefficients. If x and y are functions of t, we can determine x and y when two differential equations in x, y and t are given. We shall consider here only equations which are linear and have constant coefficients. If D denotes d/dt, such equations can be written in the form

$$f_1(D)x+f_2(D)y=\phi(t), \quad . \quad . \quad (1)$$

$$F_1(D)x+F_2(D)y=\Phi(t). \quad . \quad . \quad (2)$$

Similarly, if there are three dependent variables x, y, z, which are all functions of t, there should be three differential equations; and so on. The method of solution for three or more dependent variables will be evident from that used below for the case of two variables.

To solve equations (1) and (2), we shall obtain first an equation which contains only x. For this, operate on (1) with $F_2(D)$, on (2) with $f_2(D)$ and subtract. We get

$$\{F_2(D)f_1(D)-f_2(D)F_1(D)\}x=F_2(D)\phi(t)-f_2(D)\Phi(t).$$

This is a linear differential equation with constant coefficients which can be solved by the method of the last chapter. Let the solution be

$$x=\psi(t).$$

The substitution of this value of x in (1) or (2) will give an equation from which y can be determined. If, however, y is determined by an independent elimination, as in the case of x, the values of x and y will have to be substituted in equation (1) or (2), and the arbitrary constants in, say, y adjusted (i.e., expressed in terms of the arbitrary constants in the value of x), so that the equation may be satisfied.

Ex. Solve $\dfrac{dx}{dt}-y=t$, $\dfrac{dy}{dt}+x=1$.

These equations can be written as

$$Dx-y=t, \qquad \dots \quad (1)$$

$$x+Dy=1. \qquad \dots \quad (2)$$

Equation (1) gives $D^2x-Dy=1$. $\qquad \dots \quad (3)$

Adding (2) and (3), $D^2x+x=2$.

Hence $\quad x=c_1\cos t+c_2\sin t+(1+D^2)^{-1}2,$

or $\quad x=c_1\cos t+c_2\sin t+2.$

Substitution in (1) gives $y=-c_1\sin t+c_2\cos t-t$.

Examples

Solve the simultaneous equations

1. $\dfrac{dx}{dt}+7x-y=0$, $\dfrac{dy}{dt}+2x+5y=0$. [*Madras*, 1942]

2. $\dfrac{dx}{dt}+\dfrac{dy}{dt}-2y=2\cos t-7\sin t$,

$\dfrac{dx}{dt}-\dfrac{dy}{dt}+2x=4\cos t-3\sin t$. [*Baroda*, 1959]

3. $\dfrac{dx}{dt}=ny-mz, \dfrac{dy}{dt}=lz-nx, \dfrac{dz}{dt}=mx-ly.$ [*Baroda*, '60]

4. $Dx=2y, Dy=2z, Dz=2x$, where $D=d/dt$. [*Aligarh*, 1951]

14·3. Equations of the form $d^2y/dx^2=f(y)$.

Such equations can be solved by multiplying both sides by $2dy/dx$ and integrating with respect to x. The multiplication gives

$$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}=2f(y)\frac{dy}{dx}.$$

Integration with respect to x gives

$$\left(\frac{dy}{dx}\right)^2=c+2\int f(y)\frac{dy}{dx}dx=c+2\int f(y)\,dy.$$

By extracting the square root of both sides, separating the variables and integrating, we shall get the value of y in terms of x.

Ex. A particle moves in a straight line OA, starting from rest at A, with an acceleration towards O equal to μ times the distance of the particle from O. Find the time it will take to arrive at O.

Let P be the position of the particle at any time t after its start. Let $OP=x$. Then the equation of motion is

$$\frac{d^2x}{dt^2}=-\mu x, \quad \ldots \quad (1)$$

there being a negative sign before μx because the acceleration is towards O, i.e., in the negative direction.

Multiplying by $2dx/dt$ and integrating, we have

$$(dx/dt)^2=-\mu x^2+c. \quad \ldots \quad (2)$$

Now the particle starts at A. So, if $OA=a$, the velocity $dx/dt=0$ when $x=a$. Substitution of these values in (2) gives

$$0=-\mu a^2+c,$$

which determines c. With this value of c, (2) becomes

$$(dx/dt)^2=-\mu x^2+\mu a^2, \quad . \quad . \quad . \quad (3)$$

or

$$dx/dt=\pm\sqrt{\mu}\sqrt{(a^2-x^2)}.$$

Evidently we must choose the negative sign before the radical, because the velocity is towards O, and therefore negative.

Separating the variables and integrating,

$$t\sqrt{\mu}=c_1-\int\frac{dx}{\sqrt{(a^2-x^2)}}=c_1-\sin^{-1}(x/a). \qquad (4)$$

We know that $x=a$ when $t=0$. Hence $0=c_1-\frac{1}{2}\pi$.

With this value of c_1, (4) becomes

$$t\sqrt{\mu}=\tfrac{1}{2}\pi-\sin^{-1}(x/a).$$

Hence the value of t when $x=0$ is given by $t\sqrt{\mu}=\frac{1}{2}\pi$, i.e., the particle will arrive at O in time

$$\tfrac{1}{2}\pi/\sqrt{\mu}.$$

Examples

1. A point moves in a straight line towards a centre of force $\mu/(\text{distance})^3$, starting from rest at a distance a from the centre of force; show that the time of reaching a point distant b from the centre of force is $(a/\sqrt{\mu})\sqrt{(a^2-b^2)}$, and that its velocity then is

$$(\sqrt{\mu}/ab)\sqrt{(a^2-b^2)}.$$

2. A particle, whose mass is m, is acted upon by a force $m\mu(x+a^4x^{-3})$ towards the origin; if it starts from rest at a distance a, show that it will arrive at the origin in time

$$\tfrac{1}{4}\pi/\sqrt{\mu}.$$

Page 7

1. $\frac{1}{8}x^8,\ \frac{1}{2}x^6,\ -2x^{-1}$.
2. $\frac{3}{2}x^2+\frac{4}{3}x^3,\ 5x+7\log x$.
3. $e^x-2\cos x-3\sin x$.
4. $5\sin x+2\tan x-10x$.
5. $10^x/\log_e 10+3e^x+\frac{1}{4}x^4$.
6. $2\tan^{-1}x+3a^x/\log a$.
7. $6\sin^{-1}x+3\tan x$.
8. $\sec x+5\cot x$.
9. $-\frac{2}{3}\operatorname{cosec} x+x$.
10. $x+x^2/2!+x^3/3!+\dots$.
11. $\frac{1}{4}ax^4+\frac{1}{3}bx^3+\frac{1}{2}cx^2+dx$.
12. $-a/x+b\log x+cx$.
13. $4\log x-3/x-1/x^2$.
14. $\frac{3}{5}x^{10/3}+\frac{9}{4}x^{4/3}-21x^{1/3}$.
15. $x-16x^{-1}-\frac{64}{3}x^{-3}$.
16. $\frac{2}{7}x^{7/2}+\frac{6}{5}ax^{5/2}+2a^2x^{3/2}+2a^3x^{1/2}$.

Pages 11-12

1. $\frac{1}{4}x^4,\ \frac{1}{4}(x+2)^4,\ \frac{1}{8}(2x+3)^4,\ \frac{1}{12}(3x-2)^4,\ -\frac{1}{12}(2-3x)^4,\ (ax+b)^4/4a$.
2. $-1/2x^2,\ -1/2(x+3)^2,\ -1/6(3x+4)^2,\ 1/6(8-3x)^2,\ -1/2b(a+bx)^2,\ 1/2b(a-bx)^2$.
3. $\frac{2}{3}x^{3/2},\ \frac{2}{3}(x+4)^{3/2},\ \frac{1}{3}(2x+3)^{3/2},\ -\frac{1}{3}(3-2x)^{3/2},\ (2/3b)(a+bx)^{3/2}$.
4. $\frac{2}{5}x^{5/2},\ \frac{2}{5}(x+5)^{5/2},\ -\frac{2}{5}(5-x)^{5/2},\ \frac{1}{5}(2x-7)^{5/2},\ (2/5a)(ax-b)^{5/2}$.
5. $\log x,\ \log(x+1),\ \frac{1}{2}\log(2x+1),\ -\log(1-x),\ a^{-1}\log(ax+b),\ -a^{-1}\log(b-ax)$.
6. $\frac{1}{2}e^{2x},\ \frac{1}{2}e^{2x+3},\ -\frac{1}{2}e^{3-2x},\ a^{3x}/3\log a,\ a^{4x-5}/4\log a,\ 10^{6x}/6\log_e 10$.
7. $-\frac{1}{2}\cos 2x,\ -2\cos\frac{1}{2}x,\ -m^{-1}\cos mx,\ -m\cos(x/m),\ -\frac{1}{2}\cos(2x+\frac{1}{4}\pi)$.
8. $\sin^{-1}x,\ \frac{1}{2}\sin^{-1}2x,\ \frac{1}{2}\sin^{-1}(2x-1),\ \frac{1}{3}\sin^{-1}(3x-2),\ a^{-1}\sin^{-1}(ax+b)$.
9. $\frac{1}{2}\tan^{-1}2x,\ 2\tan^{-1}\frac{1}{2}x,\ a^{-1}\tan^{-1}ax,\ a^{-1}\tan^{-1}(ax+b),\ a\tan^{-1}(x/a),\ a^{-1}\tan^{-1}(x/a),\ (1/\sqrt{7})\tan^{-1}(x/\sqrt{7}),\ (1/2\sqrt{7})\tan^{-1}(2x/\sqrt{7}),\ (1/ab)\tan^{-1}(bx/a)$.
10. $\frac{1}{3}\sin 3x,\ -\frac{1}{5}\cot 5x,\ \frac{1}{2}\tan 2x,\ \frac{1}{3}\sin(3x+4),\ \frac{1}{7}\tan(7x+2)$.
11. $\frac{1}{9}\log(9x+1),\ 2\sin\frac{1}{2}x,\ \frac{1}{3}\tan 3x,\ \frac{1}{3}\sec 3x,\ \frac{1}{2}\sec 2x$.

12. $\frac{1}{2}\cosh 2x$, $a^{-1}\sinh(ax+b)$, $\frac{1}{3}\tanh(3x-7)$, $b^{-1}\coth(a-bx)$.

13. $-2\cos 2x-\frac{8}{3}\cos 3x$. 14. $5a^x/\log a+\frac{6}{5}a\sin(5x+2)$.

15. $-(x-\frac{1}{4})^{-2}+3(x-\frac{1}{2})^{-1}$.

16. $-3\sin\frac{1}{3}(a-x)-3\cos\frac{1}{3}(a+x)$.

Page 13

1. $\frac{1}{2}e^{x^2}$, $\frac{1}{2}a^{x^2}/\log a$, $-\frac{1}{2}\cos x^2$.
2. $\frac{1}{3}\tan^{-1}x^3$, $\frac{1}{3}(1+x^2)^{3/2}$, $\frac{1}{5}(2+x^2)^{5/2}$.
3. $(a^2+x^2)^{n+1}/2(n+1)$, $\frac{1}{2}\sin^{-1}(x^2/a)$, $(1/2a)\tan^{-1}(x^2/a)$.
4. $\frac{3}{20}(x^5-1)^{4/3}$, $\frac{2}{3}\sqrt{(a^3+x^3)}$, $-\frac{2}{3}\sqrt{(a^3-x^3)}$.
5. $(1/nb)\log(a+bx^n)$, $n^{-1}\log(4+x^n)$, $-(2/p)\sqrt{(2-x^p)}$.
6. $\frac{1}{3}\sin^3 x$, $\frac{1}{3}(\log x)^3$, $-\frac{1}{3}\cos^3 x$.
7. $\frac{1}{3}\tan^3 x$, $-\frac{1}{3}\cot^3 x$, $\frac{1}{3}(\sin^{-1}x)^3$.
8. $-\operatorname{cosec} x$, $\sec x$, $-\operatorname{cosec}^{n-1}x/(n-1)$.
9. $\tan^{p+1}x/(p+1)$, $-\frac{1}{2}(\cot^{-1}x)^2$, $\frac{1}{2}(\sec^{-1}x)^2$.
10. $\frac{1}{4}(1+\log x)^4$, $(a+be^x)^{n+1}/b(n+1)$, $\frac{1}{3}(1+\sin x)^3$.
11. $(a+b\sin x)^{p+1}/b(p+1)$, $-(a-b\tan x)^{q+1}/b(q+1)$, $(a+b\sin^{-1}x)^{m+1}/b(m+1)$.
12. $\frac{4}{3}\sin^{-1}x^3$, $\frac{4}{3}\tan^{-1}x^3$, $\frac{4}{3}\sin^{-1}(x^3/\sqrt{3})$, $\frac{2}{3}\sqrt{\frac{2}{3}}\tan^{-1}(x^3\sqrt{\frac{3}{2}})$.
13. $\sec^{-1}x$, $\frac{1}{2}\sec^{-1}x^2$; $\frac{1}{2}\sec^{-1}\frac{1}{2}x^2$.
14. $(1/2\sqrt{2})\sin^{-1}(\sqrt{2}x^2)$, $(3/2\sqrt{2})\tan^{-1}(\sqrt{2}x^2)$, $\frac{1}{28}\sec^{-1}(x^2/2)$.
15. $\frac{1}{4}(\sin^{-1}x)^4$, $\frac{1}{5}(\tan^{-1}x)^5$, $\frac{1}{6}(\sec^{-1}x)^6$, $\frac{1}{7}(\operatorname{vers}^{-1}x)^7$.

Page 15

1. $\log(x^3+1)$, $\log(x^2+3x+2)$, $\frac{1}{2}\log(ax^2+2bx+c)$, $an^{-1}\log(x^n+b)$.
2. $\log(e^x+1)$, $\log(e^x+e^{-x})$, $\log(10^x+x^{10})$.
3. $-\log(1+\cot x)$, $\log\tan^{-1}x$, $-\log\cos^{-1}x$.
4. $\log\log x$, $-b^{-1}\log(a+b\cos x)$, $\frac{1}{2}(b-a)^{-1}\log(a\cos^2 x+b\sin^2 x)$.
5. $-\tan^{-1}\cos x$, $-(1/ab)\tan^{-1}\{(b/a)\cos x\}$, $-b^{-1}\cos(a+b\log x)$.

6. $-\sin^{-1}(a^{-1}\cos x)$, $\tan(1+\log x)$, $-1/(m-1)(1+\log x)^{m-1}$.

7. $e^{\arctan x}$, $-2\cos\sqrt{x}$, $-a^{\arccos x}/\log a$, $-(1+x^2)^{-1/2}$.

8. $-\frac{1}{8}\cos^4 x^2$, $\frac{1}{20}\tan^5 x^4$.

Page 17

1. $\frac{1}{4}x^2\log(x^2/e)$, $-\log(xe)/x$, $x^{n+1}\log(x^{n+1}/e)/(n+1)^2$.

2. $e^x(x-1)$, $a^{-1}xe^{ax}-a^{-2}e^{ax}$, $x\cosh x-\sinh x$.

3. $x\sin x+\cos x$, $n^{-1}x\sin nx+n^{-2}\cos nx$, $a^{-2}(\log\sin ax-ax\cot ax)$.

4. $x\tan^{-1}x-\frac{1}{2}\log(1+x^2)$, $x\cot^{-1}x+\frac{1}{2}\log(1+x^2)$, $x\sin^{-1}x+\sqrt{(1-x^2)}$.

5. $(2-x^2)\cos x+2x\sin x$, $\frac{1}{2}(x^2-\frac{1}{2})\sin 2x+\frac{1}{2}x\cos 2x$, $m^{-3}e^{mx}(m^2x^2-2mx+2)$.

6. $x(\log x)^2-2x\log x+2x$, $x^{n+1}[(\log x)^2/(n+1)-2\log x/(n+1)^2+2/(n+1)^3]$, $-e^{-x}(x^3+3x^2+6x+6)$.

7. $\frac{1}{2}e^x(\sin x+\cos x)$, $\frac{1}{5}e^{2x}(2\sin x-\cos x)$, $\frac{1}{13}e^{3x}(2\sin 2x+3\cos 2x)$.

Page 20

1. $\log\{(x-2)/(x-1)\}$, $\frac{1}{2}\log\{(x-1)(x+3)^3\}$, $\frac{7}{2}\log\{x/(x+2)\}$.

2. $\frac{1}{4}\log\{(x-3)/(x+1)\}$, $\frac{1}{5}\log\{(x-2)^2(x+3)^3\}$, $\frac{1}{4}\log\{(x-2)/(x+2)\}$.

3. $\frac{1}{4}\sin 2x-\frac{1}{8}\sin 4x$, $\frac{1}{2}x-\frac{1}{4}\sin 2x$, $\frac{1}{12}\cos 3x-\frac{3}{4}\cos x$.

4. $-\frac{1}{12}\cos 6x+\frac{1}{4}\cos 2x$, $\frac{1}{2}x+\frac{1}{4}\sin 2x$, $\frac{3}{4}\sin x+\frac{1}{12}\sin 3x$.

Page 26

1. $2\log\tan\frac{1}{2}x+3\log\tan(\frac{1}{4}\pi+\frac{1}{2}x)$.

2. $\frac{5}{2}\log\tan(\frac{1}{4}\pi+\frac{1}{2}x^2)+7x$. **3.** $\frac{1}{2}\log\tan(\frac{1}{4}\pi+\frac{8}{2}+x)$.

4. $a^{-1}\log\tan\{\frac{1}{2}(ax+b)\}$. **5.** $\frac{1}{2}\sinh^{-1}(\frac{1}{2}x^2)$.

6. $\frac{1}{3}\cosh^{-1}(\frac{1}{3}x^3)$.

7. $\frac{1}{2}\sin x\sqrt{(4-\sin^2 x)}+2\sin^{-1}(\frac{1}{2}\sin x)$.

8. $\frac{1}{4}x^2\sqrt{(x^4+9)}+\frac{9}{4}\sinh^{-1}(\frac{1}{3}x^2)$.

9. $\frac{1}{6}x^3\sqrt{(x^6-1)}-\frac{1}{6}\cosh^{-1}x^3$.

10. $\frac{1}{2}\sec x\sqrt{(\sec^2 x+1)}+\frac{1}{2}\sinh^{-1}\sec x$.

Pages 29-30

1. $\frac{1}{11}$. **2.** 2. **3.** $\frac{5}{4}\pi$. **4.** $\frac{1}{4}\pi$. **5.** $\log 3$.
6. $1-1/\sqrt{2}$. **7.** $\frac{1}{6}\pi$. **8.** $\frac{1}{6}\pi$. **9.** $\frac{1}{3}(1-\cos a^3)$.
10. $\frac{1}{5}(1+\log 2)^5-\frac{1}{5}$. **11.** $\frac{5}{8}\pi$. **12.** $\frac{1}{192}\pi^3$.
13. $\frac{1}{4}\log(1+2/\sqrt{3})$. **14.** $\sin\log_e 3$. **15.** $\frac{1}{4}(e^2-1)$.
16. $\frac{2}{3}$. **17.** $\frac{3}{16}\pi$. **18.** 0.

Pages 34-36

1. $\frac{1}{2}(\tan^{-1}x)^2$. **2.** $\log\tan^{-1}x$. **3.** $\log\sin\log x$.
4. $m^{-1}e^{m\sin^{-1}x}$. **5.** $-\frac{1}{2}/(x^2+3)$. **6.** $\log\log\sin x$.
7. $\frac{1}{2}\tan^{-1}x^2$. **8.** $\frac{1}{2}\log\tan x$.
9. $(a/b)\log\{e^x/(b+ce^x)\}$. **10.** $\log(1-e^{-x})$.
11. $\frac{1}{4}\log\tan(2x+\frac{1}{4}\pi)$. **12.** $-x\cos x+\sin x$.
13. $\frac{1}{2}(x^2+1)\tan^{-1}x-\frac{1}{2}x$. **14.** $x\tan x+\log\cos x$.
15. $\frac{1}{2}(\sin x\cosh x-\cos x\sinh x)$.
16. $\frac{1}{4}x\sqrt{(1-x^2)}-\frac{1}{4}\sin^{-1}x+\frac{1}{2}x^2\sin^{-1}x$.
17. $(x-\tan^{-1}x)/\sqrt{(1+x^2)}$. **18.** $x-(\sin^{-1}x)\sqrt{(1-x^2)}$.
19. $\frac{1}{3}x^3\log x-\frac{1}{9}x^3$.
20. $(-n+1)^{-1}x^{-n+1}\{\log x-1/(-n+1)\}$.
21. $(x^2-2x+2)e^x$.
22. $\frac{1}{3}e^{3x}(x^2-\frac{2}{3}x+\frac{2}{9})$. **23.** $\frac{1}{3}x^3\{(\log x)^2-\frac{2}{3}\log x+\frac{2}{9}\}$.
24. $(6x/a^3-x^3/a)\cos ax+(3x^2/a^2-6/a^4)\sin ax$.
25. $\frac{1}{2}e^{x^2}(x^2-1)$.
26. $-\frac{1}{2}(\tan^{-1}x)^2+x\tan^{-1}x-\frac{1}{2}\log(1+x^2)$. **27.** $x\tan\frac{1}{2}x$.
28. $\log\{(x-2)^2/(x+1)\}$. **29.** $\log\{(x-3)^2/(x-2)\}$.
30. $\frac{1}{2}(a^2-b^2)^{-1}\log\{(x^2-a^2)/(x^2-b^2)\}$. **31.** $-1+2\log_e 2$.
32. $\frac{1}{4}\log_e\frac{32}{17}$. **33.** $2\sin x-\log\tan(\frac{1}{4}\pi+\frac{1}{2}x)$.
34. $\log\{(e^x-1)/(e^x+1)\}$. **35.** $\frac{1}{2}\cos x-\frac{1}{10}\cos 5x$.
36. $e^{\pi/2}$. **37.** $\frac{1}{5}e^{3x}\sin(4x-\tan^{-1}\frac{4}{3})$.
38. $-2\sqrt{2}\cos(\frac{1}{2}x+\frac{1}{4}\pi)$. **39.** (i) $\frac{1}{4}\pi$, (ii) $1-\frac{1}{4}\pi$, (iii) 1.
41. $e^x/(1+x)$. **42.** $\frac{1}{4}x^4/(1+x^2)^2$.
44. $x\tan^{-1}x-\frac{1}{2}\log(1+x^2)$.

45. If u_n = the given integral,

$$a^2u_n = -ax^n \cos ax + nx^{n-1} \sin ax - n(n-1)u_{n-2}.$$

46. (i) $\frac{7}{3}$, (ii) $e(e-1)$. **47.** 2.

Page 40

1. $x + \log\{(x-1)/(x+1)\}$.

2. $\frac{1}{2}x + \frac{1}{15}\log(x-1) - \frac{8}{3}\log(x+2) + \frac{27}{20}\log(2x+3)$.

3. $\frac{9}{10}\log(x+3) + \frac{4}{15}\log(x-2) - \frac{1}{6}\log(x+1)$.

4. $\frac{1}{2}\log(x+1) - 4\log(x+2) + \frac{9}{2}\log(x+3)$.

5. $\frac{1}{2}\log(x-1) + \frac{1}{18}\log(3x-1) - \frac{4}{9}\log(3x-2)$.

6. $\frac{1}{3}x^3 + x^2 + 9x - \frac{1}{6}\log(x-1) - \frac{32}{15}\log(x+2) + \frac{243}{10}\log(x-3)$.

7. $\Sigma a \log(x-a)/(a-b)(a-c)$.

8. $x + \Sigma(\alpha-a)(\alpha-b)(\alpha-c)\log(x-\alpha)/(\alpha-\beta)(\alpha-\gamma)$.

Page 42

1. $\frac{2}{9}\log\{(x-1)/(x+2)\} - \frac{1}{3}(x-1)^{-1}$.

2. $\frac{5}{27}\log\{(x-2)/(x+1)\} + \frac{1}{3}(x+1)^{-2} - \frac{4}{9}(x+1)^{-1}$.

3. $\frac{1}{3}x^{-3} + x^{-2} + 2x^{-1} + 2\log(1-1/x)$.

4. $\log\{x/(x+1)\} + 1/(x+1)$.

5. $4\log(x-2) - 3\log(x-1) - 2(x-2)^{-1} - 5(x-2)^{-2}$.

6. $\frac{1}{9}(x+2)^{-1} - \frac{2}{9}(x-1)^{-1} - \frac{1}{27}\log\{(x-1)/(x+2)\}$.

Pages 47-48

1. $(2/\sqrt{23})\tan^{-1}\{(4x+1)/\sqrt{23}\}$.

2. $\frac{1}{2}\tan^{-1}\{\frac{1}{2}(x+1)\}$.

3. $\frac{3}{4}\log(2x^2-2x+3) + \frac{1}{2}\sqrt{5}\tan^{-1}\{(2x-1)/\sqrt{5}\}$.

4. $\frac{5}{6}\log(3x^2+2x+1) - \frac{11}{6}\sqrt{2}\tan^{-1}\{(3x+1)/\sqrt{2}\}$.

5. $\frac{2}{9}\sqrt{3}\pi$. **6.** $\frac{4}{5}\sqrt{5}\log_e\{(3+\sqrt{5})/2\} - \log_e 2$.

7. $\frac{64}{17}\log\frac{5}{4} - \frac{27}{10}\log\frac{4}{3} - \frac{11}{340}\log 2 - \frac{7}{680}\pi$. **8.** π.

9. $\frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1)$

$$-(1/\sqrt{3})\tan^{-1}\{(2x+1)/\sqrt{3}\}.$$

10. $\frac{1}{4}\log\{(x+1)^2/(x^2+1)\} - \frac{1}{2}(x+1)^{-1}$.

11. $\frac{2}{3}\log(x+1) + \frac{1}{6}\log(x^2-x+1) + \frac{1}{3}\sqrt{3}\tan^{-1}\{(2x-1)/\sqrt{3}\}$.

12. $\log\{(1+x)^2/(1+x^2)\} - \tan^{-1}x$.

Page 52

1. $\frac{1}{4}x(x^2+1)^{-2}+\frac{3}{8}x(x^2+1)^{-1}+\frac{3}{8}\tan^{-1}x$.
2. $x(4x^2+2)^{-1}+\frac{1}{4}\sqrt{2}\tan^{-1}(x\sqrt{2})$.
3. $(2x+1)/3(x^2+x+1)+\frac{4}{9}\sqrt{3}\tan^{-1}\{(2x+1)/\sqrt{3}\}$.
4. $(x-3)/4(x^2+2x+3)+\frac{1}{8}\sqrt{2}\tan^{-1}\{(x+1)/\sqrt{2}\}$.
5. $\frac{1}{4}\log\{(x^2+1)/(x+1)^2\}+\frac{1}{2}(x-1)/(x^2+1)$.
6. $x-\frac{3}{2}\tan^{-1}x+\frac{1}{2}x(x^2+1)^{-1}$.
7. $\frac{3}{2}\tan^{-1}x-\frac{1}{6}\sqrt{3}\tan^{-1}(x/\sqrt{3})$.
8. $(b^2-a^2)^{-1}\{a^{-1}\tan^{-1}(x/a)-b^{-1}\tan^{-1}(x/b)\}$.
9. $\sqrt{\frac{1}{2}}\tan^{-1}\{(x^2-1)/x\sqrt{2}\}$.
10. $\frac{1}{2}\log\{(x^2-x+1)/(x^2+x+1)\}$. **11.** $\tan^{-1}(x-1/x)$.
12. $(\sqrt{2}/8a)\log\{(x^2-\sqrt{2}ax+a^2)/(x^2+\sqrt{2}ax+a^2)\}$
$\qquad +(\sqrt{2}/4a)\tan^{-1}\{(x^2-a^2)/\sqrt{2}ax\}$.
13. $(1/6\sqrt{14})\tan^{-1}\{(x^2-3)/x\sqrt{14}\}$
$\qquad -(1/6\sqrt{2})\tan^{-1}\{(x^2+3)/x\sqrt{2}\}$.
14. $\frac{1}{2}\log\{(x^2+1)/(x^2+3)\}$.
15. $\frac{1}{4}(x^2+1)^{-2}+\frac{1}{2}(x^2+1)^{-1}+\frac{1}{2}\log\{x^2/(x^2+1)\}$.
16. $\frac{1}{5}\log\{x^5/(1+x^5)\}$.
17. $\frac{1}{2}\log_e 2$. **18.** $\frac{1}{6}\log_e\frac{3}{2}+\frac{1}{3}\sqrt{2}\tan^{-1}(\frac{1}{8}\sqrt{2})$.

Page 53

1. $\frac{1}{5}\log\{(x-3)/(x+2)\}$. **2.** $\frac{1}{6}\log\{x(x-2)^3(x+3)^2\}$.
3. $x/(1-x^2)$. **4.** $\frac{1}{5}(x-1)^{-1}+\frac{1}{8}\log(x-2)-\frac{3}{25}\log(x-1)$
$\qquad -\frac{1}{400}\log(x^2+4)+\frac{7}{200}\tan^{-1}(\frac{1}{2}x)$.
5. $\frac{1}{4}(1-x)^{-1}-\frac{1}{2}\log(x-1)-\frac{1}{4}(x^2+1)^{-1}+\frac{1}{4}\log(x^2+1)$
$\qquad +\frac{1}{4}\tan^{-1}x$.
6. $\frac{1}{6}\sqrt{2}\tan^{-1}(x/\sqrt{2})-\frac{1}{6}\log\{(x-1)/(x+1)\}$.
7. $-\frac{1}{2}\tan^{-1}x+\frac{1}{4}\log\{(x-1)/(x+1)\}$.
8. $\frac{1}{8}\sqrt{2}\log\{(x^2+\sqrt{2}x+1)/(x^2-\sqrt{2}x+1)\}$
$\qquad +\frac{1}{4}\sqrt{2}\tan^{-1}\{(x^2-1)/x\sqrt{2}\}$.
9. $1/a^2x+\frac{1}{2}a^{-3}\log\{(x-a)^2(x^2+a^2)/x^4\}$.
10. $(1/n)\log\{x^n/(x^n+1)\}$.

11. $(-1/\sqrt{3})\tan^{-1}\{\sqrt{3}/(1+2x^2)\}$.

12. $-\frac{1}{4}\log(1+x^{-2}+x^{-4})-\frac{1}{2}\sqrt{3}\tan^{-1}\{(1+2x^2)/\sqrt{3}\}$.

13. $\frac{1}{18}\log\{(x^3-1)^2/(x^6+x^3+1)\}+\frac{1}{9}\sqrt{3}\tan^{-1}\{(2x^3+1)/\sqrt{3}\}$.

14. $(3x-4)/4(x-1)^2+\frac{3}{8}\log(x-1)-\frac{1}{24}\log(x+1)$
$-\frac{1}{6}\log(x^2-x+1)+\frac{1}{3}\sqrt{3}\tan^{-1}\{(2x-1)/\sqrt{3}\}$.

15. $-\frac{5}{4}(2x^2+x+2)^{-1}+\frac{14}{225}\sqrt{15}\tan^{-1}\{(4x+1)/\sqrt{15}\}$
$+7(4x+1)/60(2x^2+x+2)$.

16. $-\frac{1}{4}(x^2+a^2)^{-2}(1+x/a)-3x/8a^3(x^2+a^2)$
$-(3/8a^4)\tan^{-1}(x/a)$.

17. $-\frac{1}{16}\sqrt{2}\tan^{-1}(x/\sqrt{2})-\frac{1}{2}x(x^2+2)^{-2}-\frac{1}{8}x(x^2+2)^{-1}$.

18. $\log_e\frac{6}{5}-\frac{2}{15}$. **19.** $\frac{1}{2}\sqrt{2}\log(\sqrt{2}-1)+\frac{1}{4}\sqrt{2}\pi$.

20. $\frac{1}{2}\sqrt{2}\log(\sqrt{2}+1)+\frac{1}{4}\sqrt{2}\pi$. **21.** $\log_e\frac{4}{3}$.

22. $\frac{7}{3}\log_e 2-\frac{5}{6}\log_e\frac{7}{5}-\log_e 3$.

Page 55

1. $\frac{2}{15}(3x^2-20x+200)\sqrt{(x+5)}$. **2.** $2(\sqrt{x}-\tan^{-1}\sqrt{x})$.

3. $\frac{2}{35}(5x^3+6x^2+8x+16)\sqrt{(x-1)}$.

4. $\frac{3}{5}(x+12)(x-3)^{2/3}$.

5. $\frac{2}{5}t^5-2t^3+6t+\frac{1}{3}\sqrt{3}\log\{(t-\sqrt{3})/(t+\sqrt{3})\}$,
where $t=\sqrt{(x+2)}$.

6. $\sqrt{(x^2-a^2)}-a\tan^{-1}\sqrt{(x^2/a^2-1)}$.

7. $\frac{1}{3}\log[\{\sqrt{(1+x^3)}-1\}/\{\sqrt{(1+x^3)}+1\}]$.

8. $\frac{4}{5}t^5-t^4+\frac{8}{3}t^3-4t^2+8t-8\log(1+t)$, where $t=x^{1/4}$.

9. $-6(\frac{1}{9}t^9+\frac{1}{8}t^8+\frac{1}{7}t^7+\frac{1}{6}t^6+\frac{1}{5}t^5+\frac{1}{4}t^4)$, where $t^6=1+x$.

10. $\frac{1}{2}\log_e\frac{5}{3}$.

Page 58

1. $\sinh^{-1}\{(x+1)/\sqrt{2}\}$. **2.** $\sin^{-1}\{(2x+1)/\sqrt{5}\}$.

3. $\frac{1}{2}\sqrt{2}\sinh^{-1}\{(4x+3)/\sqrt{23}\}$.

4. $\frac{1}{8}(4x+3)\sqrt{(2x^2+3x+4)}$
$+\frac{23}{32}\sqrt{2}\log\{x+\frac{3}{4}+\sqrt{(x^2+\frac{3}{2}x+2)}\}$.

5. $\frac{1}{8}(4x+3)\sqrt{(4-3x-2x^2)}+\frac{41}{32}\sqrt{2}\sin^{-1}\{(4x+3)/\sqrt{41}\}$.

6. $\sqrt{(x^2+x+1)}-\frac{1}{2}\sinh^{-1}\{(2x+1)/\sqrt{3}\}$.

7. $2\sqrt{(x^2+3x+1)}+2\cosh^{-1}\{(2x+3)/\sqrt{5}\}$.

8. $\frac{1}{6}(2x^2+3)^{3/2}+\frac{1}{2}x\sqrt{(2x^2+3)}+\frac{3}{4}\sqrt{2}\sinh^{-1}(x\sqrt{\tfrac{2}{3}})$.

9. $\frac{1}{8}(8x^2-6x+1)\sqrt{(x^2+x+1)}-\frac{21}{16}\sinh^{-1}\{(2x+1)/\sqrt{3}\}$.

10. $\frac{1}{8}\pi(\beta-\alpha)^2$.

Pages 60-61

1. $\frac{1}{16}(4x-5)\sqrt{(2x^2-x+2)}$
$+\frac{11}{64}\sqrt{2}\log\{x-\frac{1}{4}+\sqrt{(x^2-\frac{1}{2}x+1)}\}$.

2. $\frac{1}{2}x\sqrt{(x^2+4)}-\sinh^{-1}\frac{1}{2}x$.

3. $-\frac{1}{2}x\sqrt{(3-x^2)}-\frac{1}{2}\sin^{-1}(x/\sqrt{3})$.

4. $\frac{1}{4}(2x+5)\sqrt{(x^2+x+1)}+\frac{15}{8}\sinh^{-1}\{(2x+1)/\sqrt{3}\}$.

5. $\frac{1}{3}(x^2-2)\sqrt{(x^2+1)}+3\sinh^{-1}x$. **6.** 0.

Page 62

1. $\frac{1}{2}\sqrt{2}\sinh^{-1}\{(1+x)/(1-x)\}$.

2. $\sin^{-1}\{(3x+1)/\sqrt{5}(x+1)\}$. **3.** $\frac{1}{2}\sqrt{2}\sin^{-1}\{x\sqrt{2}/(x+1)\}$.

4. $-\sqrt{(x+a)}/a\sqrt{(x-a)}$.

5. $\frac{1}{4}\sqrt{2}[\sinh^{-1}\{(1+x)/(1-x)\}+\sinh^{-1}\{(1-x)/(1+x)\}]$.

6. $\frac{1}{3}(1+x)^{1/2}(2-x)(1-x)^{-3/2}$.

7. $\sin^{-1}\{(x+3)/\sqrt{5}(x+1)\}+\cos^{-1}\{(2-x)/x\sqrt{5}\}$.

Page 65

1. $\sin^{-1}x-\sqrt{(1-x^2)}$. **2.** $\cosh^{-1}x+\sqrt{(x^2-1)}$.

3. $x+\frac{1}{2}x^2+\log x+\frac{1}{2}(x+2)\sqrt{(1+x^2)}+\frac{1}{2}\sinh^{-1}x$
$-\sinh^{-1}(1/x)$.

4. $(\frac{1}{2}x-1)\sqrt{(1-x^2)}-\frac{1}{2}\sin^{-1}x$.

5. $\frac{1}{2}[x^2-x\sqrt{(x^2-1)}+\cosh^{-1}x]$.

6. $(1/2\sqrt{33})\log[\{x\sqrt{11}+\sqrt{(3x^2-12)}\}/\{x\sqrt{11}-\sqrt{(3x^2-12)}\}]$

7. $\frac{1}{10}\sqrt{5}\log[\{\sqrt{(x^2+9)}-\sqrt{5}\}/\{(x^2+9)+\sqrt{5}\}]$
$+\frac{13}{10}\sqrt{5}\tan^{-1}\{x\sqrt{5}/2\sqrt{(x^2+9)}\}$.

8. $\frac{1}{4}\sqrt{2}\log[\{x\sqrt{2}+\sqrt{(x^2-1)}\}/\{x\sqrt{2}-\sqrt{(x^2-1)}\}]$.

Pages 70-71

1. $\frac{3}{5}x^{5/3}+\frac{45}{43}x^{43/15}+\frac{45}{61}x^{61/15}+\frac{15}{79}x^{79/15}$.

2. $\frac{3}{56}(1+x^2)^{4/3}(4x^2-3)$.

3. $(a+bx^n)^{p+1}\{(p+1)bx^n-a\}/nb^2(p+1)(p+2)$.

4. $u/3(u^3-1)+\frac{1}{9}\log\{\sqrt{(u^2+u+1)}/(u-1)\}$
$+\frac{1}{9}\sqrt{3}\tan^{-1}\{(2u+1)/\sqrt{3}\}$, where $u=x^{-1}(1+x^3)^{1/3}$.

5. $2^{-7/4}\tan^{-1}(2^{-1/4}u)+2^{-11/4}\log\{(u+2^{1/4})/(u-2^{1/4})\}$,
where $u^4=2+x^{-4}$.

6. $b^{-n-1}\sum_{p=0}^{n} {}^nC_p(a+bx^2)^{p+1-r/2}(-a)^{n-p}/(2p+2-r)$.

8. $\frac{1}{48}x(8x^4+26a^2x^2+33a^4)\sqrt{(x^2+a^2)}+\frac{5}{16}a^6\sinh^{-1}(x/a)$.

9. $\frac{5}{128}\pi a^4$.

10. $(m+2)I_m=(2m+1)aI_{m-1}$;
$-x^{m-1}(2ax-x^2)^{3/2}$, where $I_m=\int x^m\sqrt{(2ax-x^2)}\,dx$; $\frac{7}{8}\pi a^5$.

11. $\frac{1}{32}\pi a^6$.

12. If $I_{m,n}$=the given integral, the reduction formulae are
$(m+n+1)I_{m,n}=x^{m-1}(1+x^2)^{n/2+1}-(m-1)I_{m-2,n}$;
$(m+n+1)I_{m,n}=x^{m+1}(1+x^2)^{n/2}+nI_{m,n-2}$;
$(\frac{1}{13}x^4-\frac{4}{143}x^2+\frac{8}{1287})(1+x^2)^{9/2}$.

Page 72

1. $x/a^2\sqrt{(a^2-b^2x^2)}$.

2. $x/a^2\sqrt{(a^2+b^2x^2)}$.

3. $-x/b^2\sqrt{(a^2x^2-b^2)}$.

4. $(1/na^{n/2})\log[\{\sqrt{(a^n+x^n)}-a^{n/2}\}/\{\sqrt{(a^n+x^n)}+a^{n/2}\}]$.

5. $\log[\{x^2-1+\sqrt{(x^4+1)}\}/x]$.

6. $\frac{1}{2}\sec^{-1}x+\sqrt{(x^2-1)}/2x^2$.

7. $(\frac{1}{4}\pi-\frac{1}{2})a^2$.

Pages 72-73

1. $(a-b)^{-1/2}\log[\{\sqrt{(x+a)}-\sqrt{(a-b)}\}/\{\sqrt{(x+a)}+\sqrt{(a-b)}\}]$,
if $a>b$; $\{2/\sqrt{(b-a)}\}\tan^{-1}\{\sqrt{(x+a)}/\sqrt{(b-a)}\}$, if $b>a$.

2. $(\frac{4}{3}-\frac{2}{3}\sqrt{2})c^{3/2}$.

3. $4\log[\{\sqrt{(1+x)}-1\}/\sqrt{x}]-6\sqrt{(1+x)}$.

4. $(3-x)\sqrt{(3-2x-x^2)}+9\sin^{-1}\{\frac{1}{2}(1+x)\}$.

5. $\sqrt{(x^2+b^2)}+a\sinh^{-1}(x/b)$.

6. $2(a-b)^{-1}\sqrt{\{(b-x)/(x-a)\}}$.

7. $-\sqrt{\{(1-x)/(1+x)\}}$.

8. $-\sqrt{(1+1/x^2)}$.

9. $a\sin^{-1}\sqrt{(x/a)}+\sqrt{(ax-x^2)}$.

10. $\frac{2}{3}\{(1+x)^{3/2}-x^{3/2}\}$.

11. π.

12. $2\log\{\sqrt{(x-\alpha)}+\sqrt{(x-\beta)}\}$.

13. $\cosh^{-1}\{(2x+a+b)/(a-b)\}-\sqrt{\{(c-a)/(c-b)\}}\cosh^{-1}$
$[\{2(c-b)(c-a)+(a+b-2c)(x+c)\}/(a-b)(x+c)]$.

15. $\sqrt{(x^2+x+1)}-\frac{1}{2}\sinh^{-1}\{(2x+1)/\sqrt{3}\}$
$-\sinh^{-1}\{(1-x)/\sqrt{3}(1+x)\}$.

17. $2^{2n-2}\{(n-1)!\}^2/(2n-1)!$

18. $b(np+m)\int x^{m-1}(a+bx^n)^p\,dx$
$=x^{m-n}(a+bx^n)^{p+1}-a(m-n)\int x^{m-n-1}(a+bx^n)^p\,dx;$
$-\frac{1}{40}(1-x^3)^{2/3}(5x^6+6x^3+9)$.

19. $q^n(n!)/(p+1)(p+1+q)(p+1+2q)\ldots(p+1+nq)$.

Pages 82-83

1. $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x$.
2. $\sin x-\sin^3 x+\frac{3}{5}\sin^5 x-\frac{1}{7}\sin^7 x$.
3. $\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x$.
4. $-4\cos^{7/4}x\,(\frac{1}{7}-\frac{2}{15}\cos^2 x+\frac{1}{23}\cos^4 x)$.
5. $3\sin^{1/3}x\,(1-\frac{1}{7}\sin^2 x)$. 6. $\log\tan\theta+\frac{1}{2}\tan^2\theta$.
7. $\frac{1}{3}\sec^3 x-\sec x$. 8. $2\tan^{-3/2}x\,(\tan^2 x-\frac{1}{3})$.
9. $\frac{1}{2}x-\frac{1}{4}\sin 2x$.
10. $\frac{5}{16}x-\frac{5}{48}\sin x\cos x\,(3+2\sin^2 x+\frac{8}{5}\sin^4 x)$.
11. $\frac{1}{8}\sin x\,(-\cos^7 x+\frac{1}{6}\cos^5 x+\frac{5}{24}\cos^3 x+\frac{5}{16}\cos x)+\frac{5}{128}x$.
12. $\frac{1}{6}\cos x\,(\sin^5 x-\frac{1}{4}\sin^3 x-\frac{3}{8}\sin x)+\frac{1}{16}x$.
13. $(128-71\sqrt{2})/1680$. 14. $(3\pi-8)/32$. 15. $\frac{5}{32}\pi$.
16. $\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x$ 17. $\frac{1}{32}\pi$. 18. $\frac{8}{1287}$.
20. $\frac{1}{32}\pi a^6$. 22. $\frac{63}{8}\pi a^5$. 24. $\frac{1}{6}$.
26. $x^3/12(x^2+4)^{3/2}$. 27. $2-\frac{1}{2}\pi$.
28. $\frac{1}{2}\{\sin(m+n)x\}/(m+n)+\frac{1}{2}\{\sin(m-n)x\}/(m-n)$.
29. $\frac{1}{48}(2\sin 6x+3\sin 4x+6\sin 2x+12x)$.

Page 85

1. $\frac{1}{2}\tan^2 x+\log\cos x$. 2. $\frac{1}{3}\tan^3 x-\tan x+x$.
3. $-\frac{1}{3}\cot^3 x+\cot x+x$. 4. $-\frac{1}{4}\cot^4 x+\frac{1}{2}\cot^2 x+\log\sin x$
5. $\frac{1}{2}\log_e 2-\frac{1}{4}$. 6. $-\frac{1}{2}\operatorname{cosec}\theta\cot\theta+\frac{1}{2}\log\tan\frac{1}{2}\theta$.

7. $-\frac{2}{3}\cot\frac{1}{2}\theta\,(\operatorname{cosec}^2\frac{1}{2}\theta+2)$. **8.** $1/\sqrt{2}+\frac{1}{2}\log\tan\frac{3}{8}\pi$.

9. $\frac{1}{4}x(1+x^2)^{3/2}+\frac{3}{8}x(1+x^2)^{1/2}+\frac{3}{8}\log\tan(\frac{1}{4}\pi+\frac{1}{2}\tan^{-1}x)$

10. $\frac{1}{48}a^6\{67\sqrt{2}+15\log\tan\frac{3}{8}\pi\}$.

Pages 89-90

1. $\frac{2}{3}\tan^{-1}(\frac{1}{3}\tan\frac{1}{2}x)$. **2.** $\frac{2}{3}\tan^{-1}\{\frac{1}{3}\tan(\frac{1}{2}x-\frac{1}{4}\pi)\}$.

7. $\frac{1}{3}\log_e 2$. **8.** $(1/\sqrt{2})\tan^{-1}\{(\tan x)/\sqrt{2}\}$.

9. $\frac{1}{2}\tan^{-1}(2\tan x)$.

10. $\{1/a\sqrt{(a^2-b^2)}\}\tan^{-1}\{a\tan x/\sqrt{(a^2-b^2)}\}$.

11. $x/b-(a/b)\int dx/(a+b\cos x)$. Now apply § 4·3.

12. $-2\sqrt{(1-\sin x)}-\sqrt{2}\log\tan(\frac{1}{4}x+\frac{1}{8}\pi)$.

13. $-1/2(1+2\tan x)$. **14.** $(1/ab)\tan^{-1}\{(b/a)\tan x\}$.

15. $(a^2+b^2)^{-1/2}\log\tan\frac{1}{2}\{x+\tan^{-1}(b/a)\}$.

16. $\frac{18}{25}x+\frac{1}{25}\log(3\sin x+4\cos x)$.

18. $\{a\theta+b\log(a\cos\theta+b\sin\theta)\}/(a^2+b^2)$.

19. $\frac{1}{3}\log\{(1+\cos x)\sin x/(1+2\cos x)^2\}$.

20. $\frac{1}{5}\log\{(3+2\cos x)^2\sin\frac{1}{2}x\sec^5\frac{1}{2}x\}$.

21. $\frac{1}{2}\log\tan\frac{1}{2}x+\tan\frac{1}{2}x+\frac{1}{4}\tan^2\frac{1}{2}x$.

22. $x\cos\alpha+\sin\alpha\log\sin(x-\alpha)$.

23. $-(b-a)^{-1/2}\sin^{-1}\{\sqrt{(1-a/b)}\cos x\}$ if $a<b$.

24. $-2b^{-2}\{\log(a+b\cos x)+a/(a+b\cos x)\}$. **25.** $\frac{5}{27}\pi$.

Pages 92-93

1. $(\frac{1}{4}-\frac{1}{2}x^2)\cos 2x+\frac{1}{2}x\sin 2x$. **2.** $\frac{1}{4}x^2-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x$.

4. $\frac{1}{8}\pi a^2(1+\frac{1}{6}\pi^2)$. **5.** $\frac{5}{16}\pi^4-15\pi^2-120$.

6. $\frac{1}{4}e^{2x}\{(1/\sqrt{13})\cos(3x-\tan^{-1}\frac{3}{2})+(3/\sqrt{5})\cos(x-\tan^{-1}\frac{1}{2})\}$

7. $\frac{1}{4}e^{ax}\{3(1+a^2)^{-1/2}\sin(x-\cot^{-1}a)$
$-(a^2+9)^{-1/2}\sin(3x-\cot^{-1}\frac{1}{3}a)\}$.

8. $\frac{1}{2}e^x\{x(\cos x+\sin x)-\cos x\}$. **9.** $\frac{3}{25}$. **10.** $\frac{24}{85}$.

Pages 95-97

1. $-\frac{1}{3}x+\frac{1}{4}\log(e^x-1)+\frac{1}{12}\log(e^x+3)$.

2. $-1/(1+e^x)$. **3.** $x-\log(e^x-1)-1/(e^x-1)$.

4. $e^{m\cos^{-1}x}\{x-m\sqrt{(1-x^2)}\}/(1+m^2)$.

5. $\frac{1}{2}e^{m\tan^{-1}x}[1/m+\{1/\sqrt{(m^2+4)}\}\cos\{2\tan^{-1}x$
$-\tan^{-1}(2/m)\}]$.

6. $e^x/(x+2)$. **7.** $e^x(x-1)/(x+1)$.

8. $e^x(x+1)/(x+2)$. **10.** $x\log(1+x^2)-2x+2\tan^{-1}x$.

11. $\{x\sin^{-1}x+\sqrt{(1-x^2)}\}(\log x-1)-\sqrt{(1-x^2)}$
$+\log[\{1+\sqrt{(1-x^2)}\}/x]$.

12. $x\log\{x+\sqrt{(x^2-a^2)}\}-\sqrt{(x^2-a^2)}$.

13. $\log x-(1+1/x)\log(1+x)$. **14.** $e^x\log x$.

15. $\frac{1}{2}\{\log(\sec x+\tan x)\}^2$. **16.** $\frac{1}{6}\pi-\frac{2}{9}$.

17. 0. **18.** $\pi^2/12$. **19.** $\frac{1}{5}x^5\{(\log x)^2-\frac{2}{5}\log x+\frac{2}{25}\}$.

20. $\frac{1}{3}x^3\log(1-x^2)-\frac{2}{9}x^3-\frac{2}{3}x-\frac{1}{3}\log\{(1-x)/(1+x)\}$.

21. $x-\frac{1}{3}x^3/1!+\frac{1}{5}x^5/2!-\frac{1}{7}x^7/3!+\dots$.

22. $mx-\frac{1}{3}m^3x^3/3!+\frac{1}{5}m^5x^5/5!-\dots$.

23. $\log x+1/2(3!)x^2-1/4(5!)x^4+\dots$.

24. $x-\frac{1}{12}x^3-\frac{1}{480}x^5+\dots$.

25. $(a+x)\tan^{-1}\sqrt{(x/a)}-\sqrt{(ax)}$.

26. $\sqrt{(e^{2x}+ae^x)}+\frac{1}{2}a\log\{e^x+\frac{1}{2}a+\sqrt{(e^{2x}+ae^x)}\}$.

27. $\frac{1}{2}\sec^{-1}\cosh x+\frac{1}{2}\operatorname{sech}x\tanh x$.

28. $\frac{1}{2}\sin x\cosh x+\frac{1}{2}\cos x\sinh x$. **29.** $\cosh x\tan\frac{1}{2}x$.

30. $\int\tanh^n x\,dx=\tanh^{n-1}x/(1-n)+\int\tanh^{n-2}x\,dx$.

32. $\{\frac{1}{2}-1/(1+x)\}\tan^{-1}x+\frac{1}{4}\log\{(1+x)^2/(1+x^2)\}$.

33. $-\frac{1}{5}(\cos\log x+2\sin\log x)/x^2$.

34. $-(\cos^{-1}x)/2x^2+\frac{1}{2}\sqrt{(1/x^2-1)}$.

35. $x\sin^{-1}x/\sqrt{(1-x^2)}+\frac{1}{2}\log(1-x^2)$.

36. $x(\sin^{-1}x)^2+2\sqrt{(1-x^2)}\sin^{-1}x-2x$.

37. $-2\sinh^{-1}\sqrt{(\cos x)}$. **38.** $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}$.

39. $\frac{1}{2}\sin 2x\log(1+\tan x)+\frac{1}{2}\log\sin(x+\frac{1}{4}\pi)-\frac{1}{2}x$.

40. $\log\log\tan x$. **41.** $-2\operatorname{cosec}\alpha\sqrt{\{\sin(x+\alpha)/\sin x\}}$.

42. $\cos\alpha\cos^{-1}(\cos x\sec\alpha)-\sin\alpha\cosh^{-1}(\sin x\operatorname{cosec}\alpha)$.

43. $(b^2+c^2)^{-1/2}\sinh^{-1}\{\sqrt{(b^2+c^2)}\tan\theta/\sqrt{(a^2+c^2)}\}$.

44. $\frac{1}{2}$.

Pages 97-100

1. $-\frac{1}{3}\cos^3 x+\frac{2}{5}\cos^5 x-\frac{1}{7}\cos^7 x$. **2.** $\frac{1}{8}x-\frac{1}{32}\sin 4x$.

3. $2(\frac{1}{5}-\frac{1}{9}\sin^2 x)\sin^{5/2}x$.

4. $\tan x-2\cot x-\frac{1}{3}\cot^3 x$.

5. $\frac{8}{693}$. **6.** $\frac{5}{192}\pi$. **7.** $\frac{3}{512}\pi$. **8.** $\frac{1}{16}\pi$.

9. $\frac{1}{16}(\frac{1}{4}\pi-\frac{1}{3})/a^3$. **10.** $4^{n-1}\{(n-1)!\}^2/(2n-1)!\,a^{2n}$.

11. $(\frac{9}{32}\pi-\frac{23}{35})a^7$. **12.** $-\frac{1}{4}+\frac{1}{2}\log_e 2$.

14. $-\frac{2}{3}\sin x\cos x/(1+2\cos x)^2$
$$-\tfrac{1}{9}\sqrt{3}\log\{\cos(\tfrac{1}{2}x+\tfrac{1}{6}\pi)/\cos(\tfrac{1}{2}x-\tfrac{1}{6}\pi)\}.$$

15. $\frac{1}{5}x^5+\frac{1}{21}x^7+\frac{2}{135}x^9+\ldots$.

16. $\operatorname{cosec}(a-b)\log\{\sin(x-a)/\sin(x-b)\}$.

17. $2x-3\tan^{-1}(1+\tan^{-1}\frac{1}{2}x)$.

18. $c\theta+b\log(1+\sin\theta)-2(a-c)/(1+\tan\frac{1}{2}\theta)$.

19. $x(ap+bq)/(a^2+b^2)$
$$+\{(aq-bp)/(a^2+b^2)\}\log(a\sin x+b\cos x).$$

20. $\frac{1}{4}\pi$. **21.** $(u-e\sin u)/(1-e^2)^{3/2}$, where $\sqrt{(1-e)}\tan\frac{1}{2}\theta$
$$=\sqrt{(1+e)}\tan\tfrac{1}{2}u.$$

24. $\frac{2}{27}-\frac{1}{12}\pi^2$. **25.** $(\sin x-x\cos x)/(x\sin x+\cos x)$.

26. $\alpha(\alpha^2+\beta^2+\gamma^2)/\{(\alpha^2+\beta^2+\gamma^2)^2-4\beta^2\gamma^2\}$. **28.** $e^x\tan\frac{1}{2}x$.

29. $I_{m,n}=(1+mI_{m-1,\,n-1})/(m+n)$. **32.** $l^{r+1}(x)$.

Pages 104-105

1. $\frac{4}{3}$. **2.** $\frac{5}{16}\pi$. **3.** $\frac{8}{3}$. **4.** $\pi^2/2ab$.

5. $\frac{2}{3}\pi$. **10.** $\frac{3}{512}\pi^2$. **11.** $\frac{1}{2}\pi\log_e 2$. **14.** $\pi\log_e 2$.

Pages 110-111

1. $\frac{3}{2}$. **2.** e^b-e^a. **3.** $\sin b-\sin a$. **4.** $\cos a-\cos b$.

5. $2(\sqrt{b}-\sqrt{a})$. **6.** $1/a-1/b$. **7.** $\frac{1}{3}$.

8. $\log_e 2$. **9.** $\frac{3}{8}$. **10.** $\frac{1}{4}\pi$. **11.** $(2k)!/(2^k k!)^2$.

12. $\frac{1}{2}\pi$. **13.** $\frac{1}{3}\log_e 2$. **15.** $\frac{1}{2}\tan 1$.

16. $4/e$. **18.** e^{-1}.

Pages 117-119

2. $(2n)!\,\pi/(2^n.n!)^2$. **7.** $\frac{1}{4}\pi$.

9. $\frac{1}{2}\pi \log_e 2$. **11.** $2/a$. **13.** $\frac{1}{4}(b^4-a^4)$.
15. $\frac{31}{14}$. **16.** $\frac{1}{2}\pi+1$. **18.** $\pi a^n(1-a^2)$.

Pages 119-124

1. $\frac{2}{21}x^{3/2}(7-3x^2)$.
2. $\log\{x+\sqrt{(x-1)}\}-\frac{2}{3}\sqrt{3}\tan^{-1}[\{2\sqrt{(x-1)}+1\}/\sqrt{3}]$.
3. $x\cos^{-1}(1/x)-\cosh^{-1}x$. **4.** $e^x+\log(e^x-1)$.
5. $\frac{1}{2}e^x(x-1)+\frac{1}{10}e^x\{\cos(2x-2\tan^{-1}2)$
$-\sqrt{5}x\cos(2x-\tan^{-1}2)\}$.
6. $x\tan x+\log\cos x-\frac{1}{2}x^2$. **7.** $1/x-\tan(1/x)$.
8. $-\frac{1}{2}\log\tan(\frac{1}{4}\pi+x)$. **9.** $2\sqrt{(\sin^{-1}x)}$.
10. $(1/\sqrt{13})\,e^{2x}\cos(3x+\frac{1}{3}\pi-\tan^{-1}\frac{3}{2})$.
11. $2\sinh^{-1}(\frac{1}{2}\sin\frac{1}{2}x)$.
12. $-\sqrt{(4x-x^2)}+3\sin^{-1}(\frac{1}{2}x-1)$.
13. $\frac{1}{3}x^3\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\log(1+x^2)$.
14. $e^{m\tan^{-1}x}(m+x)/(1+m^2)\sqrt{(1+x^2)}$. **15.** $-2/\sqrt{(\sin x)}$.
16. $\frac{2}{5}(5+\tan^2x)\sqrt{(\tan x)}$. **17.** $\frac{1}{120}(1+4/x^2)^{3/2}(1-6/x^2)$.
18. $x/c\sqrt{(ax^2+c)}$. **19.** $\frac{1}{2}\sqrt{2}\sin^{-1}\{x\sqrt{2}/(1+x^2)\}$.
20. $-e^x\cot\frac{1}{2}x$. **21.** $\log(\tan x\tan\frac{1}{2}x)$.
22. $\frac{1}{3}\log\{(1+\sin x)/(2-\sin x)\}$.
24. $(-x^4+12x^2-24)\cos x+(4x^3-24x)\sin x$.
25. $(a^2-b^2)^{-3/2}\int(a-b\cos\theta)\,d\theta$;
$2(a^2-b^2)^{-1/2}\tan^{-1}\sqrt{\{(a-b)/(a+b)\}}$;
$\{1/b(a^2-b^2)^{3/2}\}[2(a^2-b^2)\tan^{-1}\sqrt{\{(a-b)/(a+b)\}}$
$+b\sqrt{(a^2-b^2)}-a^2\cos^{-1}(b/a)]$.
26. $A=b/(n-1)(b^2-a^2)$; $B=-(2n-3)a/(n-1)(b^2-a^2)$;
$C=(n-2)/(n-1)(b^2-a^2)$.
$2a(a^2-b^2)^{-3/2}\tan^{-1}[\sqrt{\{(a-b)/(a+b)\}}\tan\frac{1}{2}x]$
$-b\sin x/(a^2-b^2)(a+b\cos x)$ if $a>b$; or
$-2a(b^2-a^2)^{-3/2}\tanh^{-1}[\sqrt{\{(b-a)/(b+a)\}}\tan\frac{1}{2}x]$
$+b\sin x/(b^2-a^2)(a+b\cos x)$ if $b>a$.
28. π. **30.** $-\cot x\log(\cos x+\sqrt{\cos 2x})-\cot x-x$
$+\operatorname{cosec}x\sqrt{(\cos 2x)}$.

32. $\frac{3}{2}-\frac{1}{4}\pi$.

33. $\frac{1}{2}\operatorname{cosech}\alpha\log\{(1+\cosh\alpha+\sinh\alpha)/(1+\cosh\alpha-\sinh\alpha)\}$.

36. $cA+aC=2bB$, or $B^2=AC$.

38. $\pi\sqrt{\pi}a^{n+1}\,\Gamma(\frac{1}{2}n+1)/2\Gamma(\frac{1}{2}n+\frac{3}{2})$. **39.** $\pi\log_e 2$.

41. If the given integral be denoted by I_n, then
$I_n=-\{1/(2n+1)\}x^{2n}(1-x^2)^{1/2}+\{2n/(2n+1)\}I_{n-1}$;
$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\{(2n-1)/(2n)\}\frac{1}{2}\pi$. **44.** $-\frac{181}{105}$.

48. $2\pi\{a-\sqrt{(a^2-b^2)}\}/b^2$. **54.** $\frac{1}{3}\pi$.

Pages 131-132

1. e^b-e^a. **2.** $c^2\sinh(a/c)$. **3.** $b\log(b/e)-a\log(a/e)$.
4. $\frac{1}{4}\pi$. **8.** πab. **10.** $\frac{32}{105}b^{7/2}a^{-3/2}$.
11. πa^2. **12.** $\frac{4}{3}a^2$. **14.** $4a^2$. **15.** πa^2.
16. $\frac{9}{32}$. **17.** $\frac{1}{3}(9\sqrt{3}+4\pi)a^2$. **19.** $\frac{3}{8}\pi ab$.

Pages 135-136

1. $\frac{1}{2}a^2\log(\beta/\alpha)$. **2.** $\frac{1}{4}l^2(\tan\frac{1}{2}\alpha+\frac{1}{3}\tan^3\frac{1}{2}\alpha)$.
3. $a^2(e^{2m\beta}-e^{2m\alpha})/4m$. **4.** $\frac{1}{16}\pi a^2$.
5. $(4\pi+3\sqrt{3})/(2\pi-3\sqrt{3})$. **6.** $\frac{1}{2}\pi(a^2+b^2)$.
7. $\frac{1}{96}\pi a^2(\pi^2-6)$. **8.** $\frac{3}{2}\pi a^2$.
9. 11π. **10.** $\frac{1}{3}\pi$. **14.** $\frac{1}{2}\pi a^2$. **16.** $(\pi-1)a^2$.

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2. $3\sqrt{3}a^2/2$.

Pages 142-143

2. 0·1982. **3.** 7·78. **4.** 0·84. **5.** 710 sq. ft. **7.** 0·82.

Pages 143-146

2. $\frac{3}{8}\pi a^2$. **4.** $\frac{1}{2}a^2(4-\pi)$. **7.** $\frac{16}{3}a^2$. **9.** $3\pi a^2$.
12. $\frac{1}{12}\pi(a^2+b^2)$. **16.** $\frac{5}{4}\pi a^2$. **17.** $(15\sqrt{3}/16-\frac{1}{2}\pi)a^2$.
18. $(9\pi+16)/(9\pi-16)$. **21.** $\frac{1}{8}a^2(3\sqrt{3}-4)$. **23.** $\frac{3}{2}\pi c^2$.

Pages 149-151

1. $a\{\sqrt{2}+\log(1+\sqrt{2})\}$. **2.** $f(y_1)-f(a)$, where
$f(y)=\log[y/\{1+\sqrt{(1+y^2)}\}]+\sqrt{(1+y^2)}$.

4. $x\sqrt{(x^2+\frac{1}{4})}+\frac{1}{4}\log\{x+\sqrt{(x^2+\frac{1}{4})}\}+\frac{1}{4}\log_e 2$; 1·48.

6. $\log(e+e^{-1})$. **9.** $8a$. **11.** $4a(\cos\frac{1}{2}\alpha-\cos\frac{1}{2}\beta)$.
12. $8a$. **13.** $f(r_2/a)-f(r_1/a)$, where $f(\theta)=\frac{1}{2}a[\theta\sqrt{(1+\theta^2)}+\log\{\theta+\sqrt{(1+\theta^2)}\}]$.
14. πa. **15.** 768π. **16.** $8a$.
17. $f(\theta_2)-f(\theta_1)$, where $f(\theta)=a[\sqrt{(3+\sec^2\theta)}-\sqrt{3}\log\{\sqrt{(1+3\cos^2\theta)}+\sqrt{3}\cos\theta\}]$.

Pages 153-154

5. The evolute of $y^2=4ax$ is $27ay^2=4(x-2a)^3$.
6. $s=4a\sin\frac{1}{3}\psi$.

Pages 154-157

5. $\sqrt{2}(e^{\pi/2}-1)$.
15. $s=\frac{1}{2}a\{\theta\sqrt{(1+\theta^2)}+\sinh^{-1}\theta\}$, $\psi=\theta+\tan^{-1}\theta$.
16. $s=a\{\tan\psi\sec\psi+\log(\tan\psi+\sec\psi)\}$.

Pages 162-164

2. $\pi h^2(a-\frac{1}{3}h)$. **4.** $2\pi ah^2$. **5.** $\frac{4}{3}\pi ab^2$.
6. $(2\pi b/3a)\{6a^2b-b^3-3ab\sqrt{(a^2-b^2)}-3a^3\sin^{-1}(b/a)\}$.
11. 4π; $4\pi^2$. **12.** $\frac{32}{105}\pi a^3$.

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3. $\frac{128}{1215}\pi a^2(125\sqrt{10}+1)$. **4.** $\frac{8}{3}\pi a^2(2\sqrt{2}-1)$.
5. $8\pi(1+4\pi/3\sqrt{3})$.

Pages 169-170

1. $6\sqrt{2}\pi^2a^3$, where a=semi-major axis. **2.** $8\sqrt{2}\pi a^3/15$.
4. $\frac{11}{2}\pi^2a^3$. **5.** (i) $(4a/3\pi,\ 4a/3\pi)$; (ii) $(2a/\pi,\ 2a/\pi)$, where a=radius of the circle.

Pages 170-174

2. $\frac{2}{15}\pi$. **3.** $2\pi a^3(\log_e 2-\frac{2}{3})$. **5.** $\frac{32}{15}\pi$.
6. $1536\pi/5$ cu. in. **8.** 90π. **11.** $\frac{4}{5}$.
13. π^2, $2\pi^2$. **16.** $\frac{1}{5}$, $\frac{5}{28}\pi$. **20.** $A=\frac{518}{225}a$, $\frac{259}{450}\pi^2a^2$.
22. $\frac{1}{12}\pi a^3\{(3/\sqrt{2})\log(1+\sqrt{2})-1\}$, $\pi^2a^3/4\sqrt{2}$.
26. $\pi a^2\{3\sqrt{2}-\log(1+\sqrt{2})\}$.
27. $\pi a^2[\sqrt{6}-1-(1/\sqrt{2})\log\{(2+\sqrt{3})/(1+\sqrt{2})\}]$.

28. $48\sqrt{2}\pi a^2/5$. **29.** $\frac{8}{3}\pi a^3$, $\frac{32}{5}\pi a^2$. **33.** Volume and surface between $x=a$ and $x=b$ are $c\{F(b)-F(a)\}$ and $\{2F(b)-F(a)\}$, where $F(x)=\frac{1}{4}\pi c\{c\sinh(2x/c)+2x\}$.
35. $\frac{8}{3}\pi a^3$.

Pages 181-182

1. $\xi=\frac{1}{4}a[t(1+2t^2)\sqrt{(1+t^2)} -\log\{t+\sqrt{(1+t^2)}\}]/[t\sqrt{(1+t^2)}+\log\{t+\sqrt{(1+t^2)}\}]$;
$\eta=\frac{4}{3}a\{(1+t^2)^{3/2}-1\}/[t\sqrt{(1+t^2)}+\log\{t+\sqrt{(1+t^2)}\}]$.
2. $\xi=x-a(y-a)/s$, $\eta=\frac{1}{2}y+\frac{1}{2}ax/s$, where s is the length of the arc.
3. $\xi=\eta=\frac{2}{5}a$.
4. $\xi=\frac{2}{3}a(\sin\alpha)/\alpha$, $\eta=0$; where the angle subtended at the centre by the arc is 2α and the axis of x is the middle radius.
5. $\xi=\frac{2}{3}a(\sin^3\alpha)/(\alpha-\sin\alpha\cos\alpha)$, $\eta=0$, where the symbols have the same meaning as in the previous example; $\xi=4a/3\pi$.
6. $\xi=\frac{1}{2}\pi$, $\eta=\frac{1}{8}\pi$. **7.** $\xi=\frac{5}{7}b$. **8.** $\xi=\frac{5}{8}a$.
9. $\xi=\frac{1}{3}a(3\pi-8)/(4-\pi)$. **10.** $\xi=\frac{5}{6}a$. **11.** $\xi=\frac{1}{8}\pi a\sqrt{2}$.
12. $\xi a^{1/3}=\eta b^{1/3}=\frac{9}{20}a^{2/3}b^{2/3}$. **13.** $\xi=\frac{8}{5}a/m^2$, $\eta=2a/m$.
14. $\xi=50a/63$. **15.** $\xi=\frac{2}{3}h$. **16.** $\xi=\frac{4}{5}a$.

Pages 184-185

1. $\xi=\frac{2}{3}$(length of vertical side).
2. At middle point of median which bisects the horizontal base.
3. $\xi=\frac{3}{4}$(depth of base).
4. $\xi=\frac{2}{3}(h_1^2+h_1h_2+h_2^2)/(h_1+h_2)$.
5. $\xi=\frac{1}{2}(3h_2^2+2h_2h_1+h_1^2)/(2h_2+h_1)$.
6. $\xi=\frac{1}{4}\{3\pi(a^2+4b^2)+32ab\}/(4a+3\pi b)$. **7.** $\xi=h+a^2/4h$.
8. $\xi=\frac{1}{7}(35h^2-42ah+15a^2)/(5h-3a)$, where $a=\frac{1}{4}$ latus rectum.
10. Its depth is $\frac{55}{54}$ times that of the centre.

Pages 189-190

1. $\frac{2}{5}Mc^2$, where $2c$ is the equatorial diameter.
2. $\frac{4}{5}ahM$. 3. $\frac{1}{2}Ma^2$.
4. $Ma^2\{1+\frac{1}{2}\cos 2\alpha-(3/4\alpha)\sin 2\alpha\}$, where 2α is the angle subtended by the arc at the centre.
5. $\frac{35}{36}Ma^2$. 6. $\frac{1}{2}Ma^2$, where a=length of rod.

Pages 190-192

1. At $2a/\pi$ from centre.
2. $(0, \frac{2}{3}a)$. 3. $(n+1)(b^{n+2}-a^{n+2})/(n+2)(b^{n+1}-a^{n+1})$.
4. $\frac{1}{8}a(7+3\log_e 2)$, $\frac{1}{32}a(21+16\log_e 2)$.
6. Mid. pt. of radius perp. to base.
7. If $\rho=y/a$, mass$=\frac{1}{3}a^2$, $\xi=\frac{3}{8}a$, $\eta=\frac{3}{16}a\pi$. 8. $\xi=\frac{3}{8}a$.
9. Depth of C.P.$=(d^2+dh+\frac{1}{3}h^2)/(d+\frac{1}{2}h)$.
10. $\frac{1}{2}Ma^2$, if length of each side$=2a$. 11. $\frac{1}{24}Ma^2$.
12. (i) $\xi=\eta=\frac{2}{5}a$, (ii) $\frac{144}{35}Ma^2$. 14. $\frac{1}{2}M(R^2+r^2)$.
15. $\frac{7}{5}Ma^2$ foot^2-lbs. 17. $\frac{2}{3}Ma^2$.
20. $W(b-\frac{1}{3}b^3/a^2)$ inch-tons.
21. $12^2 . 2^n . 100(3^{1-n}-2^{1-n})/(1-n)$ foot-lbs.;
$12^2 . 400(1-\sqrt{2}/\sqrt{3})$ foot-lbs.

Page 199

1. $xy=ce^{y-x}$. 2. $\log\{x(1-y)^2\}=c+\frac{1}{2}x^2-\frac{1}{2}y^2-2y$.
3. $y-x=c(1+xy)$. 4. $y=c(1-ay)(x+a)$.
5. $\tan x\tan y=c$. 6. $y+\frac{2}{3}(x-2a)\sqrt{(a+x)}=c$.
7. $y\sin y=x^2\log x+c$. 8. $e^y=e^x+\frac{1}{3}x^3+c$.

Page 201

1. $\log(y-x)=c+x/(y-x)$.
2. $e^{\arctan(y/x)}\sqrt{(x^2+y^2)}=c$.
3. $cx^2=y+\sqrt{(x^2+y^2)}$. 4. $(x-y)^2=cxy^2$.
5. $c(x-y)^{2/3}(x^2+xy+y^2)^{1/6}=$
$\exp[(1/\sqrt{3})\tan^{-1}\{(x+2y)/x\sqrt{3}\}]$, where $\exp x\equiv e^x$.
6. $x^2+y^2=cx$.

7. $2xy(x+y)^{-2}+\log(x+y)=c$.
8. $x/y+\log(xy)=c$.
9. $x(x^2+xy+y^2)=c$.
10. $x^2-y^2=c(x^2+y^2)^2$.
11. $y^3=ce^{x^3/y^3}$.
12. $xy\cos(y/x)=c$.

Page 204

1. $\tan^{-1}\{(2y+1)/(2x+1)\}=\log\{c\sqrt{(x^2+y^2+x+y+\frac{1}{2})}\}$.
2. $x+y-2=c(y-x)^3$.
3. $\tan^{-1}\{(y-2)/(x-3)\}+\log[c\sqrt{\{(y-2)^2+(x-3)^2\}}]=0$.
4. $x+y+\frac{4}{3}=ce^{3(x-2y)}$.
5. $4x+8y+5=ce^{4(x-2y)}$.
6. $\frac{2}{7}(2x+3y)-\frac{9}{49}\log(14x+21y+22)=x+c$.
7. $x+2y-5=c(2x-y)^2$.
8. $\frac{3}{2}(x^2+y^2)+2xy-5(x+y)=c$.

Pages 206-207

1. $xy=\frac{1}{3}x^3+\frac{3}{2}x^2+2x+c$.
2. $y=c(x+a)^3+\frac{1}{2}(x+a)^5$.
3. $y=e^{mx}/(m+a)+ce^{-ax}$.
4. $ye^x=c-e^x/x+\int e^x x^{-1}\,dx$.
5. $y=ce^{nx}-m(nx+1)/n^2-q/n$.
6. $y\sqrt{(1+x^2)}=c+\frac{1}{2}\log\tan(\frac{1}{2}\tan^{-1}x)$. Another form is $y\sqrt{(1+x^2)}=c+\frac{1}{2}\log[\{\sqrt{(1+x^2)}-1\}/x]$.
7. $y=c(1-x^2)+\sqrt{(1-x^2)}$.
8. $(x^2+1)y=\frac{4}{3}x^3+c$.
9. $(x-1)y=x^2(x^2-x-c)$.
10. $xy=c-\tan^{-1}x$.
11. $x^2y=c+(2-x^2)\cos x+2x\sin x$.
12. $y=ce^{\sin x}-(1+\sin x)$.
13. $y=\cos x+c\sec x$.
14. $(1+x^2)y=c+\sin x$.
15. $(\frac{1}{3}+y)\tan^3\frac{1}{2}x=c+2\tan\frac{1}{2}x-x$.
16. $y=\tan x+c\sqrt{\tan x}$.
17. $y=c\cos x+\sin x$.
18. $y=cx+x\log\tan x$.
19. $x=ce^{-\arctan y}+\tan^{-1}y-1$.
20. $x=y-a^2+ce^{-y/a^2}$.
21. $x=y^3+cy$.

22. $(x-2y^3)y^2=c.$ **23.** $y=ce^{-x/\sqrt{(1-x^2)}}+x(1-x^2)^{-1/2}.$

24. $3(1+x^2)y=4x^3.$ **25.** $4xy=x^4+3.$

Page 209

1. $xy\log(c/x)=1.$ **2.** $x=y(1+c\sqrt{x}).$

3. $cy=(1-y)\sqrt{(1-x^2)}.$

4. $(\log ex+cx)y=1.$ **5.** $e^{-x^2/2}=(c+\cos x)y.$

6. $1/xy=c-\int x^{-1}\sin x\,dx.$

7. $\sqrt{(1+x^2)}=(c+\sinh^{-1}x)y.$

8. $(x+1)^2y^3=\frac{1}{6}x^6+\frac{2}{5}x^5+\frac{1}{4}x^4+c.$

9. $y^{-2}=ce^{x^2}+1+x^2.$ **10.** $y^{-2}e^{x^2}=2x+c.$

11. $y^{-2}=-1+(c+x)\cot(\frac{1}{2}x+\frac{1}{4}\pi).$

12. $x^3y^{-3}=3\sin x+c.$ **13.** $cx^5y^5+\frac{5}{2}x^3y^5=1.$

14. $y^{-1}e^x=c-x^2.$ **15.** $y^{-1}=c\cos x+\sin x.$

Pages 211-212

1. $xy(ax+by)=c.$ **2.** $x^2+y^2+2a^2\tan^{-1}(x/y)=c.$

3. $x^3-3axy+y^3=c.$ **4.** $(e^y+1)\sin x=c.$

5. $2(x+y)+\sin 2x+\sin 2y-4\sin\alpha\sin x\sin y=c.$

6. $y(x+\log x)+x\cos y=c.$ **7.** $x+ye^{x/y}=c.$

8. $ax^2+2hxy+by^2+2gx+2fy+c=0.$

Page 217

1. $x^2-y-1-x\cos y=cx.$ **2.** $ax^2y-cy+2e^x=0.$

3. $\frac{1}{2}x^2y^2+\log(x/y)-1/xy=c.$

4. $\log(x^2/y)-1/xy=c.$ **5.** $cy\cos xy=x.$

6. $x^4y(3+y^2)+x^6=c.$ **7.** $6x^7y+3x^4y^2-x^6=c.$

8. $e^{6y}\{\frac{1}{2}x^2y^2-\frac{1}{3}x^3+\frac{1}{6}y^2-\frac{1}{18}y+\frac{1}{108}\}=c.$

9. $3x^2y^4+6xy^2+2y^6=c.$ **10.** $x^2y^3(1+2xy)=c.$

11. $x^3y^2+4x^2y^6=c.$ **12.** $4x^{1/2}y^{1/2}-\frac{2}{3}x^{-3/2}y^{3/2}=c.$

Page 218

1. $y^2+x\log cx=0.$ **2.** $y^2=3x^2-6x-x^3+ce^{-x}+4.$

3. $x\log y=e^x(x-1)+c.$ **4.** $\sin y=(e^x+c)(1+x).$

5. $cx^2+2xe^{-y}=1.$ **6.** $y=\tan\frac{1}{2}(x+y)+c.$

7. $y=a\tan^{-1}\{(x+y)/a\}+c.$

8. $4x+y+1=2\tan(2x+c).$

9. $e^y=c\exp(-e^x)+e^x-1.$

10. $\sqrt{(x^2+y^2)}=a\sin\{c+\tan^{-1}(y/x)\}.$

Pages 218-221

1. $y^{-2}+\sin^2 x+\sin x+\frac{1}{2}=ce^{2\sin x}.$

2. $x+y=ce^{x-y}.$

3. $y^2=x(x+c).$

4. $y^2(x+ce^x)=1.$

5. $cx=e^{x/y}.$

6. $y=x\sinh(x+c).$

7. $x-2y+\frac{1}{2}\log(x^2+y^2)=c.$

8. $x^3y=\frac{1}{6}x^6+c.$

9. $y=ce^{-\tan x}+\tan x-1.$

10. $y(1+x^2)=\tan^{-1}x-\frac{1}{4}\pi.$

11. $\log(x/y)-1/xy=c.$

12. $(x+y)^3=3a^2x+x^3+c.$

13. $(x^2+y^2)^2+2a^2(y^2-x^2)=c.$

14. $y(1+x^3)=\frac{1}{2}x-\frac{1}{4}\sin 2x+c.$

15. $y=1+ce^{1/x}.$

16. $x(x^2y^2+\cos xy)=c.$

17. $y^{-1}\sec^2 x=c-\frac{1}{3}\tan^3 x.$

18. $6x^3y^3-27xy^4+3y^3\log y-y^3=c.$

19. $xy\sqrt{(x^2-y^2)}=c.$

20. $1/y^2(1-x^2)=c-1/(1-x^2)-\log(1-x^2).$

21. $(b-a)\log\{(x+y)^2-ab\}=2(x-y)+c.$

22. $xy=c\cos x+\sin x.$

23. $xe^{\arctan y}=\arctan y+c.$

24. $a\log\{(x-y-a)/(x-y+a)\}=2y+c.$

25. $y(1+\sqrt{x})/(1-\sqrt{x})=x+\frac{4}{3}x^{3/2}+c.$

26. $x^2+y^2=c(x+y).$

27. $3y^2=-2x^2e^{-1/x^3}+cx^2.$

28. $\frac{1}{2}e^{2y}=\frac{1}{3}e^{3x}+\frac{1}{3}x^3+c.$

29. $y^2=x^2+cx-1.$

30. $x^4y^3+4x^3y^5=c.$

31. $xy=\sin x-x\cos x+c.$

32. $xy+y^2-3y=x^2+x+c.$

33. $y^{1-n}=ce^{(n-1)\sin x}+2\sin x+2/(n-1).$

34. $y(1+x)e^{-x}=xe^{-x}+c.$

35. $2\tan y=ce^{-x^2}+x^2-1.$

36. $\frac{2}{3}x^3y-\frac{1}{2}y^3+e^x=cy.$

37. $\tan y=c(1-e^x)^3.$

38. $x=ce^y-y-2.$

39. $x+y-4\log(2x+3y+7)=c.$

40. $xy/(x-1)=\frac{1}{3}x^3+c.$

41. $16x^2y=4x^4\log x-x^4+c.$

42. $\frac{13}{12}x^{-36/13}y^{24/13}-\frac{13}{5}x^{-10/13}y^{-15/13}=c.$

46. $y=\cos x-2\cos^2 x.$ **47.** $x^4+y^4+6x^2y^2=c.$
48. $(x+y+1)^3+cxy=0.$ **49.** $(2x+y+1)(x+y)=c.$
50. $y^7 \tan x=ce^{-y}.$

Pages 223-224

1. $(y-3x+c)(y+x+c)=0.$
2. $(y+x-1+ce^{-x})(2xy+x^2+c)(y+x^2+c)=0.$
3. $(x^3-3y+c)(e^{x^2/2}+cy)(xy+cy+1)=0.$
4. $(y-c)(xy+cy-1)(y-ce^{1/x})=0.$ **5.** $y(1\pm\cos x)=c.$
6. $(y+x+1-ce^x)(2y+x^2-c)=0.$
7. $(y-x+c)(xy+c)=0.$
8. $(y-x+c)(x^2+y^2-c^2)=0.$ **9.** $(y-cx^2)(yx^3-c)=0.$

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1. $y=3x-a\log(\frac{1}{3}-ce^{3x/a}).$
2. $\frac{1}{2}\log(p^2/x^2-p/x+1)+(1/\sqrt{3})\tan^{-1}\{(2p-x)/x\sqrt{3}\}$
$=\log(c/x)$, with the given relation.
3. $\cos[\{\sqrt{(1-c^2+2cx-x^2)}-y\}/(c-x)]=c-x.$
4. $x=b\log p+2cp+A$, with the given relation.
5. $x=a\log\{y+\sqrt{(y^2-a^2)}\}+c.$ **6.** $xy=c+c^2x.$
7. $yp^2=\frac{3}{2}a+2bp+cp^2$, with the given relation.
8. $(2x-b)c=y^2-ac^2.$ **9.** $\log y=cx+c^2.$

Page 228

1. $y=cx-ac(c-1).$ **2.** $y=cx+(1+c^2)^{1/2}.$
3. $c=\log(cx-y).$ **4.** $(y-cx)(c-1)=c.$
5. $xc^2-yc+a=0.$ **6.** $(c+1)(y^2-cx^2)+ch^2=0.$
7. $y^2=cx+\frac{1}{8}c^3.$ **8.** $y^2=cx^2+c^2.$

Page 229

1. $y=cx+c^3.$
2. $x=(\log p-p+c)(p-1)^{-2}$, with the given relation.
3. $x=cp^{-2}-\{n/(n+1)\}p^{n-1},\ y=2cp^{-1}-\{(n-1)/(n+1)\}p^n.$
4. $\sin^{-1}(y/x)=\pm\log cx.$

5. $p^n x = n-1+c\exp\{p^{-n+1}/(n-1)\}$, with the given relation.
6. $c^2y^2+2cx=1$. 7. $x^2+y^2=cx$.
8. $y^2=cx^2-bc/(ac+1)$.
9. $c^2(x^2-a^2)-2cxy+y^2+a^2=0$.
10. $x=cp^{a/(1-a)}+\{3b/(2-3a)\}p^2$, with the given relation.
11. $(y-cx^2)(y^2+3x^2-c)=0$.
12. $y^2-x^2\sinh^2(c+x)=0$.
13. $y=4c(cxy+1)$. 14. $y=2c\sqrt{x}+f(c^2)$.
15. $xp=\frac{1}{3}p^3+c$, with the given relation.
16. $x+2p-2=ce^{-p}$, with the given relation.
17. $(y+c)^2+(x-a)^2=1$.
18. $y(1-p^2)^{1/2}+(1-p^2)^{3/2}=c$, with the given relation.
19. $y=cx-ac^2/(c+1)$.
20. $(c-y)(1+p)^2=1$, with the given relation.
21. $e^y=ce^x+c^3$. 22. $(y-cx)^2/(1+c^2)=a^2$.
23. $(cx-y)^2=c^2-1$; $x^2-y^2=1$.
24. $x=(2c+3p^2-2p^3)/2(p-1)^2$; $y=(2cp^2+2p^3-p^4)/2(p-1)^2$.

Page 233

1. $x^4y''+n^2y=0$. 2. $y^2=x^2+2xy\,dy/dx$.
3. $yy''+y'^2=0$. 5. $xyy''+xy'^2=yy'$.
6. The graph consists of the system of circles $x^2+y^2=a^2$, where a is arbitrary. 7. $x-y=\tanh x$.

Page 240

1. $(y-cx)^2+a^2c=0$; $4xy=a^2$.
2. $(3y+c)^2=2cx^3$; $6y=x^3$. 3. $y=c(x-c)^2$; $27y=4x^3$.
4. $y-cx=ac/\sqrt{(1+c^2)}$; $x^{2/3}+y^{2/3}=a^{2/3}$.
5. $y=cx+ac/(c-1)$; $\sqrt{x}+\sqrt{y}=\sqrt{a}$.
6. $y=cx+\sqrt{(b^2+a^2c^2)}$; $x^2/a^2+y^2/b^2=1$. The complete

primitive is a system of straight lines, each member touching the ellipse $x^2/a^2+y^2/b^2=1$.

7. $(y+c)^2=x(x-1)(x-2)$; $x=0$, $x=1$, $x=2$.
8. $(y-cx)^2=m^2+c^2$, $y^2+m^2x^2=m^2$.

Pages 241-242

1. $y=ke^{ax}$. 2. $1/r=b\theta+c$. 3. $x^2-y^2=c^2$.
5. $y^2-x^2-2xy(dy/dx-\tan\alpha)/(1+\tan\alpha\, dy/dx)+c=0$.
6. $y/a=\cosh(x/a+c)$. 7. $1/r=k-ae^{\theta}$.
9. $y=ce^{-x/ky}$. 10. $y^2=x+5$.
13. $u=c_1\tan^{-1}x+c_2$; 0 when n is even, and $(-1)^{(n-1)/2}(n-1)!$ when n is odd.

Pages 245-246

1. $x^2+2y^2=c^2$. 2. $2x^2=2y-1+ce^{-2y}$.
3. $y^2=ce^{x^2+y^2}$. 4. $y^2-x^2=c^2$.
6. $y=yp^2+2xp$. 7. $(x^{3/2}+c^{3/2})^2=\frac{9}{4}ay^2$.
9. $r^2=c^2e^{\theta^2}$. 10. $r=ce^{-\theta^2/2}$.
11. $(\log r)^2+\theta^2=c^2$. 12. $r^{n^2}=c(1-\cos n\theta)$.
13. $r^n=c^n\cos n\theta$. 14. $r^n\cos n\theta=c^n$.
15. $r=ce^{-\theta/\sqrt{3}}$. 16. $(x^2+y^2)^{1/2}=ce^{\tan^{-1}(y/x)}$.
17. $x^2+y^2+c\sqrt{3}x+cy=0$.

Pages 246-248

1. $D^3y+D^3y(Dy)^2=3Dy(D^2y)^2$. 2. $y(Dy)^2+2xDy=y$.
5. $cx=y^k$. 6. $r=ce^{k\theta}$.
7. $x^2+y^2=Ae^{2kx}-x/k-1/2k^2$. 8. $r=k\sin(\theta+c)$.
9. $c\pm x=a\log\tan\{\frac{1}{2}\sin^{-1}(y/a)\}+(a^2-y^2)^{1/2}$.
10. $x^2-y^2=k^2$. 11. $y=a\log\sec(x/a+c)+b$.
12. $y^2=4c(x+c)$.
13. $yy'=k(x^2+y^2)$, $x^2+y^2=ae^{2kx}-x/k-1/2k^2$.
14. $c^2x-cxy+a^3=0$, $xy^2=4a^3$.

15. $c^2-2cy^2-4cxy+4x^2y^2=0, y=0$ and $y+4x=0$.
16. $y^2=(x+c)^3$, $y=0$. **17.** Circles with centres on the given line and passing through the given point.
20. $2x^2+3y^2=c^2$. **21.** $xy'(x^2+y^2-1)=y(x^2+y^2+1)$; $(x^2+y^2)(x+yy')=x-yy'$; $(x^2+y^2)^2-2(x^2-y^2)=c$.
24. $4y+2x+\sin 2x=c$. **25.** $r=2b/(1-\cos\theta)$.

Pages 252-253

1. $ae^x+be^{-x}-\frac{1}{2}\sin x$; ae^x+be^{-x}; $-\frac{1}{2}\sin x$.
2. $y=c_1e^{-x}+c_2e^{-2x}$. **3.** $y=c_1e^{3x}+c_2e^{4x}$.
4. $y=c_1e^{-x}+c_2e^{-4x}$. **5.** $y=c_1e^x+c_2e^{2x}+c_3e^{3x}$.
6. $y=c_1e^x+c_2e^{3x}+c_3e^{-4x}$. **7.** $y=c_1e^x+c_2e^{3x}+c_3e^{5x}$.
8. $x=0$.

Page 259

1. $y=(c_1+c_2x)e^x$. **2.** $y=(c_1+c_2x)e^{3x}$.
3. $y=c_1e^{-px}\cos(qx+c_2)$.
4. $y=c_1e^{2x}+c_2\cos(2x+c_3)$.
5. $y=c_1e^{2x}+c_2e^{-x/2}\cos(\frac{1}{2}\sqrt{3}x+c_3)$.
6. $y=c_1\cos(x+c_2)+(c_3+c_4x)e^x$.
7. $y=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x$.
8. $y=c_1e^{mx/\sqrt{2}}\cos(mx/\sqrt{2}+c_2)$
$+c_3e^{-mx/\sqrt{2}}\cos(mx/\sqrt{2}+c_4)$.
9. $y=c_1\exp(\mu x)+c_2\exp(-\mu x)$; $y=c_3\cos(\mu x+c_4)$.
10. $y=2\sqrt{2}\cos(x+\frac{1}{4}\pi)$.

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1. $y=c_1e^{-x/2}\cos(\frac{1}{2}x\sqrt{3}+c_2)+e^{-x}$.
2. $y=c_1e^{-px}\cos(qx+c_2)+e^{ax}/\{(p+a)^2+q^2\}$.
3. $y=c_1e^x+c_2e^{2x}+\frac{1}{12}e^{5x}$.
4. $y=c_1e^{-15x}+c_2e^{-16x}+\frac{136}{105}e^{-x}$.
5. $y=(c_1+c_2x)e^{kx}+e^x/(k-1)^2$.
6. $y=c(e^x-e^{6x})+\frac{1}{4}e^x(1-e^x)$.

Pages 270-271

1. $y=c_1\cos(x+c_2)-\frac{1}{3}\cos 2x.$
2. $y=c_1e^{-x/2}\cos(\frac{1}{2}\sqrt{3}x+c_2)-\frac{2}{13}\cos 2x-\frac{3}{13}\sin 2x.$
3. $y=c_1e^{4x}\cosh(x\sqrt{7}+c_2)+\frac{25}{29}\cos 5x-\frac{10}{29}\sin 5x.$
4. $y=c_1e^{x}\cos(2x+c_2)+\frac{3}{26}\cos 3x-\frac{1}{13}\sin 3x.$
5. $y=c_1e^{x}+c_2e^{2x}-\frac{1}{130}(7\cos 3x+9\sin 3x).$
6. $y=c_1e^{2x}\cosh(\sqrt{3}x+c_2)+\frac{1}{73}a(8\cos 2x-3\sin 2x).$
7. $x=-ae^{-nt\cos\alpha}\{\sin(nt\sin\alpha)\}/(n^2\sin 2\alpha)$
$$+a\sin nt/(2n^2\cos\alpha).$$
8. $y=c_1\cos(3x+c_2)+\frac{1}{5}(\cos 2x+\sin 2x).$
9. $y=c_1\cos(2x+c_2)-\frac{1}{4}x\cos 2x+\frac{1}{5}e^{x}.$
10. $y=c_1e^{2x}+c_2e^{-2x}-\frac{1}{3}e^{x}-\frac{1}{8}\sin 2x.$

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1. $y=c_1+c_2e^{2x}\cos(x+c_3)+\frac{2}{5}x.$
2. $y=(c_1e^{2x}+c_2e^{3x}+\frac{1}{6}x+\frac{5}{36}.$
3. $y=(c_1+c_2x)e^{x}+c_3e^{-x}+x+1.$
4. $y=c_1e^{-x}\cos(x+c_2)+\frac{1}{2}(x-1).$
5. $y=c_1e^{2x}+c_2e^{-2x}-\frac{1}{4}x^2-\frac{1}{8}.$
6. $y=(c_1+c_2x)e^{2x}+\frac{1}{4}(x^2+2x+\frac{3}{2}).$
7. $y=c_1+c_2e^{-x}+c_3e^{-2x}+\frac{1}{12}x(2x^2-9x+21).$
8. $y=c_1e^{2x}+(c_2+c_3x)e^{-x}-\frac{1}{4}(2x^2-6x+9).$
9. $y=c_1e^{-2x}+c_2\cos(2x+c_3)+\frac{1}{16}(2x-1).$
10. $y=c_1e^{x}+c_2e^{-2x}-\frac{1}{10}(\cos x+3\sin x)-\frac{1}{4}(2x+1).$
11. $y=c_1\cos(2x+c_2)-\frac{1}{8}(x\sin 2x-1).$
12. $y=c_1+(c_2+c_3x)e^{-x}+\frac{1}{18}e^{2x}+\frac{1}{6}x(2x^2-9x+24).$

Pages 276-277

1. $y=c_1e^{x}+c_2e^{2x}-\frac{1}{2}xe^{x}(x+2).$
2. $y=c_1e^{2x}\cosh(x\sqrt{3}+c_2)-\frac{1}{7}e^{2x}\sin 2x.$
3. $y=c_1e^{x}\cos(\sqrt{3}x+c_2)+\frac{1}{3}e^{x}\cos x.$

4. $y=c_1e^x+c_2e^{-x}+c_3\cos(x+c_4)-\frac{1}{5}e^x\cos x$.
5. $y=(c_1+c_2x)e^x+\frac{1}{8}e^{3x}(2x^2-4x+3)$.
6. $y=(c_1+c_2x)e^x+\frac{1}{2}x^2e^x$. 7. $y=e^x(2-x)+e^{2x}$.
8. $y=c_1e^x+c_2e^{-x}+\frac{1}{5}(2\sinh x\sin x-\cosh x\cos x)$.
9. $y=e^x(c_1+x)+c_2e^x\cos(x+c_3)+\frac{1}{10}(3\sin x+\cos x)$.
10. $y=e^{-3x/2}(c_1+c_2x+18x^2)$.
11. $y=e^{-2x}(c_1+c_2x-\frac{1}{2}x^2)+\frac{1}{16}e^{2x}$.
12. $y=e^x(c_1+c_2x+c_3x^2+\frac{1}{24}x^4+\frac{1}{6}x^3)$.
13. $y=c_1e^{-x}+c_2e^{-2x}+c_3e^{3x}-\frac{1}{12}e^{2x}(x+\frac{17}{12})$.
14. $y=c_1e^{-x/2}\cos(\frac{1}{2}\sqrt{3}x+c_2)+c_3e^{x/2}\cos(\frac{1}{2}\sqrt{3}x+c_4)$
$+ax^2-2a-\frac{1}{481}be^{-x}(9\sin 2x+20\cos 2x)$.
15. $y=e^{2x}(c_1+\frac{1}{16}x^2-\frac{9}{64}x)+c_2e^{-6x}$.

Page 280

1. $y=(c_1+c_2x)e^x+\frac{1}{2}(x\cos x+\cos x-\sin x)$.
2. $y=c_1+c_2e^{-x}+\frac{1}{2}x(\sin x-\cos x)+\cos x+\frac{1}{2}\sin x$.
3. $y=c_1\cos(x+c_2)-\frac{8}{9}x\cos 2x+\frac{1}{27}(26-9x^2)\sin 2x$.
4. $y=-x^3\cos x+3x^2\sin x$.
5. $y=c_1e^x+c_2e^{-x}-\frac{1}{2}(x\sin x+\cos x)+\frac{1}{12}xe^x(2x^2-3x+9)$.

Pages 280-282

1. $y=c_1e^{-ax}+c_2e^{-bx}$. 2. $y=(c_1+c_2x)e^{ax}$.
3. $y=(c_1+c_2x)e^{-x/2}\cos(\frac{1}{2}x\sqrt{3})+(c_3+c_4x)e^{-x/2}\sin(\frac{1}{2}x\sqrt{3})$.
4. $y=c_1\cos(3x+c_2)+c_3\cos(2x+c_4)$.
5. $y=c_1e^{2x}+c_2e^{-6x}+\frac{1}{9}e^{3x}$.
6. $y=c_1e^{-3x}+c_2e^{-2x}+\frac{1}{12}e^x$.
7. $y=c_1e^{-7x/3}+e^{2x}(c_2+x)$.
8. $y=(c_1+c_2x+\frac{1}{2}x^2)e^{-x/2}$.
9. $y=c_1e^x+c_2e^{-x}+\frac{1}{2}x\sinh x$.
10. $y=c_1e^t+c_2e^{2t}+\sin t+3\cos t$.
11. $y=c_1e^{-x}+c_2e^{2x}+\frac{1}{20}(\cos 2x-3\sin 2x)$.

12. $y=(c_1+c_2x)e^{-kx}+A\{(k^2-p^2)\cos px+2kp\sin px\}/(k^2+p^2)^2.$

13. $y=c_1\cos ax+(c_2+x/2a)\sin ax.$

14. $y=c_1e^{-x}\cos(3x+c_2)+6\cos 3x-\sin 3x,\ y=1.$

15. $y=c_1e^x+c_2e^{12x}+\frac{1}{144}(12x+13).$

16. $y=c_1e^{-\sqrt{7}x/2}\cos(\frac{3}{2}x+c_2)+c_3e^{\sqrt{7}x/2}\cos(\frac{3}{2}x+c_4)+x^2+\frac{127}{8}.$

17. $y=c_1e^x\cos(\sqrt{2}x+c_2)+\frac{1}{4}(\cos x-\sin x)+\frac{1}{27}(9x^2+12x+2).$

18. $y=(c_1+c_2x+\frac{1}{8}x^2)e^x+(c_3+c_4x)\sin x$
$+(c_5+c_6x)\cos x-\frac{1}{32}x^2\sin x+\frac{1}{2}.$

19. $y=(c_1+\frac{1}{8}x)e^x+c_2e^{-x}+c_3e^{-x/2}\cos(\frac{1}{2}\sqrt{7}x+c_4)$
$-\frac{1}{4}(2x^2-2x+3).$

20. $y=c_1+(c_2+3x)e^x+c_3e^{-x}+c_4\cos(x+c_5)+x^2+2x\sin x.$

21. $y=e^{-x/2}[(\frac{1}{4}x+c_1)\cos(x\sqrt{3}/2)$
$+(c_2+\frac{1}{12}\sqrt{3}x)\sin(x\sqrt{3}/2)]+c_3e^{x/2}\cos(x\sqrt{3}/2+c_4).$

22. $y=e^{-x}(c_1+c_2x+c_3x^2+\frac{1}{6}x^3).$

23. $y=(c_1-\frac{1}{2}x)e^x+c_2e^{2x}+\frac{1}{12}e^{-x}.$

24. $y=c_1\cos(x+c_2)+\frac{1}{2}e^{-x}+\frac{1}{2}x\sin x+x^3-6x$
$-\frac{1}{5}e^x(2\cos x-\sin x).$

25. $y=c_1e^{-x}+c_2e^x+c_3\cos(x+c_4)+\frac{1}{8}(x^2\cos x-3x\sin x).$

26. $y=(c_1+c_2x)e^x-e^x(2\cos x+x\sin x).$

27. $y=(c_1+c_2x+3\sin 2x-2x^2\sin 2x-4x\cos 2x)e^{2x}.$

28. $y=\frac{1}{12}(6ax^2+2bx^3+cx^4+12d).$ **30.** $y=e^x-1.$

Pages 287-288

1. $y=(c_1+c_2\log x)/x^2.$

2. $y=c_1+c_2\log x+\frac{1}{2}a(\log x)^2.$

3. $y=c_1/x+x(c_2+c_3\log x)+\frac{1}{4}x^{-1}\log x.$

4. $y=c_1x^2+c_2x^3+\frac{1}{2}x.$ **5.** $y=c_1+c_2/x+x^2.$

6. $y=c_1x^2+c_2x^{-2}+\frac{1}{4}x^2\log x.$

7. $y=c_1x+c_2\cos(\sqrt{3}\log x+c_3)+\frac{1}{7}x^2+\frac{1}{4}x\log x.$

8. $y=c_1x+c_2/x+\frac{1}{3}x^2.$ **9.** $y=c_1x+c_2x^2+1/6x.$

10. $y=c_1+c_2/x+\frac{1}{2}(\log x)^2-\log x.$

11. $y=c_1\cos(\log x^2+c_2)/x^3+\frac{1}{13}\log x-\frac{6}{169}$.

12. $y=c_1/x+c_2\sqrt{x}\cos(\frac{1}{2}\sqrt{3}\log x+c_3)+\frac{1}{2}x+\log x$.

13. $y=c_1x\cos(\log x+c_2)+x\log x$.

14. $y=c_1/x+c_2x^3-\frac{1}{3}x^2\log x-\frac{2}{9}x^2$.

Pages 289-290

1. $x=e^{-6t}(c_1\cos t+c_2\sin t)$,

$y=e^{-6t}\{(c_1+c_2)\cos t-(c_1-c_2)\sin t\}$.

2. $x=3\cos t+c_1e^{\sqrt{2}t}+c_2e^{-\sqrt{2}t}$,

$y=2\sin t+c_1(1+\sqrt{2})e^{\sqrt{2}t}-c_2(\sqrt{2}-1)e^{-\sqrt{2}t}$.

3. $x=a_1\sin kt+a_2\cos kt+a_3$, $y=b_1\sin kt+b_2\cos kt+b_3$. $z=c_1\sin kt+c_2\cos kt+c_3$, where $k^2=l^2+m^2+n^2$; and the arbitrary constants are connected by the following relations :

$$(mc_1-nb_1)/a_2=(na_1-lc_1)/b_2=(lb_1-ma_1)/c_2=k,$$
$$la_1+mb_1+nc_1=0,\ a_3/l=b_3/m=c_3/n.$$

4. $x=c_1e^{2t}+c_2e^{-t}\cos(c_3+\sqrt{3}t)$.

$y=c_1e^{2t}-c_2e^{-t}\cos(c_3-\frac{1}{3}\pi+\sqrt{3}t)$,

$z=c_1e^{2t}-c_2e^{-t}\cos(c_3+\frac{1}{3}\pi+\sqrt{3}t)$.

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